Analytical approach to Boolean PCSPs

with Katzper Michno

Boolean PCSPs

$$\mathrm{PCSP}(\mathbb{A}, \mathbb{B}) \quad \text{where } A = B = \{0, 1\}$$

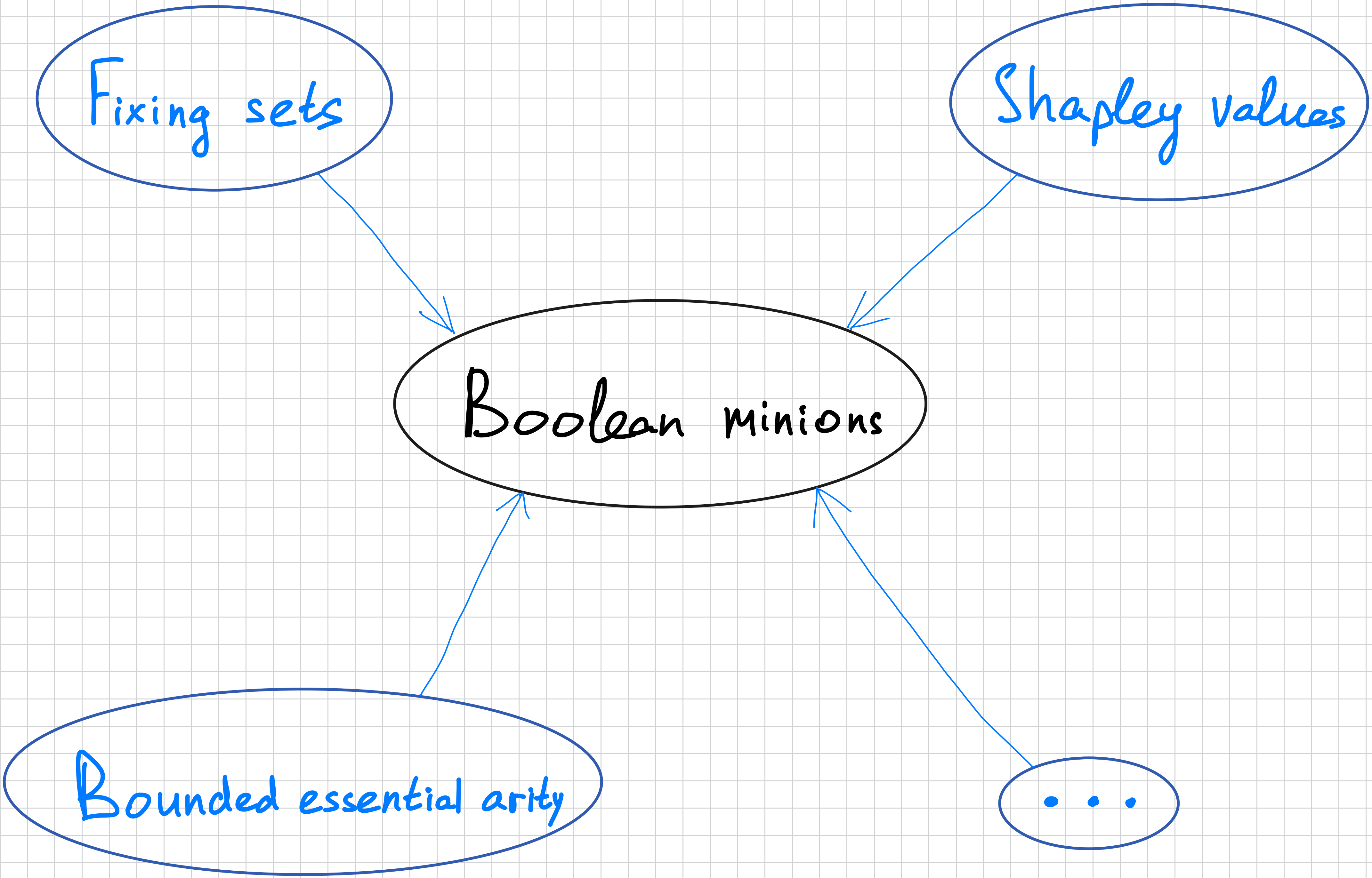
$$\mathrm{Pol}(\mathbb{A}, \mathbb{B}) \subseteq \{ f : \{0,1\}^n \to \{0,1\} \mid n \in \mathbb{N} \}$$

Algorithms:

$$\max(x_1 \dots x_n) \qquad \min(x_1 \dots x_n)$$

$$\mathrm{thr}_t(x_1 \dots x_n) = \begin{cases} 1 & \sum x_i > tn \\ 0 & \text{else} \end{cases}$$

$$\mathrm{at}(x_1 \dots x_n\, y_1 \dots y_{n+1}) = \begin{cases} 1 & \sum y_i > \sum x_j \\ 0 & \text{else} \end{cases}$$


Fixing sets
Shapley values
Boolean minions
Bounded essential arity
. . .

Ordered PCSPs (BGS'21)

$$\mathbb{A} = (\{0,1\};\ \leq, \dots)$$
$$\mathbb{B} = (\{0,1\};\ \leq, \dots)$$

$$\begin{array}{c} f(x_1 \dots x_n) = \cdot \\ \rotatebox{90}{$\leq$} \ \cdots \ \rotatebox{90}{$\leq$} \ \Rightarrow \ \rotatebox{90}{$\leq$} \\ f(x_1' \dots x_n') = \cdot \end{array}$$

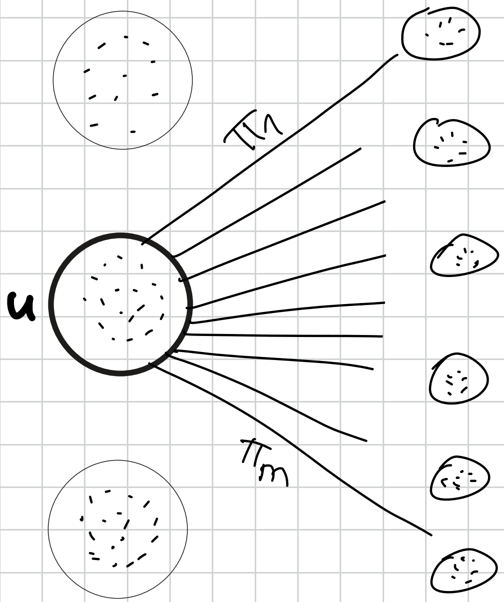
in particular $\forall f \in \mathrm{Pol}(\mathbb{A}, \mathbb{B})$ f is monotone

Th (BGS'21) $\forall$ Ordered PCSP$(\mathbb{A}, \mathbb{B})$ is either

- in P or
- NP-hard, assuming Rich 2-to-1 Conjecture

Rich 2-to-1 Conjecture

Input: Label Cover instance



- all constraint maps $\pi : [2n] \to [n]$ are 2-to-1

- $\forall u$ each $\pi \in \{\pi : [2n] \to [n] \mid \text{2-to-1}\}$ is incident equally often

$\forall \varepsilon\ \exists n$ s.t. NP-hard to distinguish

YES: possible to satisfy all edges

NO: not possible to satisfy ε-fraction

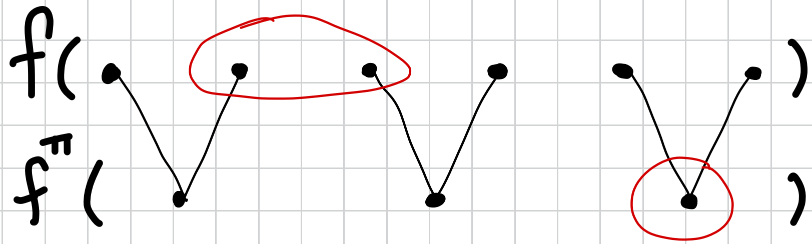
Rich 2-to-1 Conjecture

Algebraic perspective: If

$\exists C, \varepsilon > 0$ and $\mathrm{Sel}: \mathrm{Pol} \to \mathcal{P}(\mathbb{N})$ s.t. $\forall f \in \mathrm{Pol}$
$f: A^n \to B$

- $\mathrm{Sel}(f) \subseteq [n]$ and $|\mathrm{Sel}(f)| < C$
- $\Pr_{\pi}\left[\pi(\mathrm{Sel}(f)) \cap \mathrm{Sel}(f^{\pi}) \neq \emptyset\right] \geq \varepsilon$
  where $\pi \in_R \{\pi: [2n] \to [n] \mid \text{2-to-1}\}$

then NP-hard, assuming Rich 2-to-1 Conjecture



Ordered PCSPs (BGS'21)

(Shapley value)

Monotone $f(x_1 \dots x_n)$: $\forall_{i=1..n}\ \varphi_i(f) \in [0,1]$ & $\sum_i \varphi_i(f) \leq 1$

Th (BGS'21) $\forall$ Ordered PCSP($\mathbb{A}, \mathbb{B}$) is either

- in P ($\Leftrightarrow \forall \varepsilon \exists f \in \mathrm{Pol}$ s.t. $\max_i \varphi_i(f) < \varepsilon$) or
- NP-hard, assuming Rich 2-to-1 Conjecture ($\Leftrightarrow \exists \varepsilon \forall f \in \mathrm{Pol}\ \max_i \varphi_i(f) > \varepsilon$)

then $\mathrm{Sel}(f) := \{ i \mid \varphi_i(f) > \tau \}$

Basics

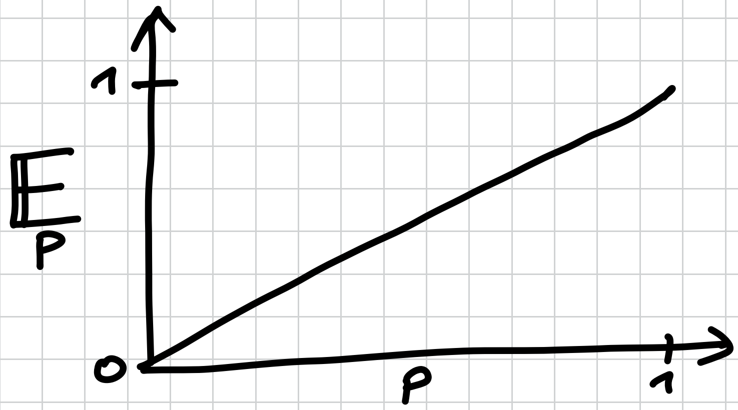
Fix $p \in [0,1]$ and $f: \{0,1\}^n \to \{0,1\}$

Let $\forall_i \quad x_i = \begin{cases} 1 & \text{with } \Pr = p \\ 0 & \text{with } \Pr = 1-p \end{cases}$

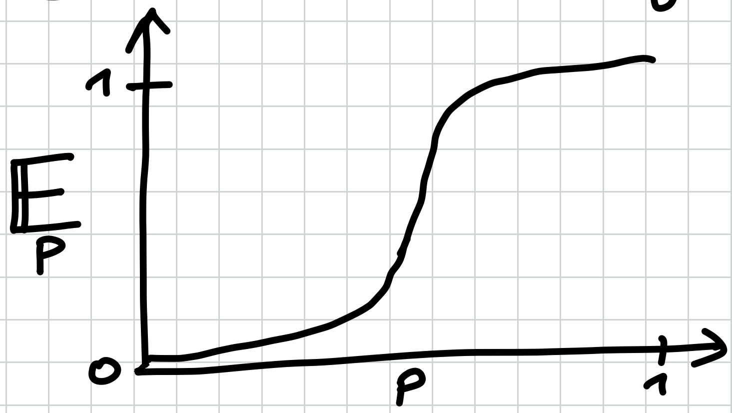
$$\mathbb{E}_p[f] := \mathbb{E}_{x_i}[f(x_1 \ldots x_n)]$$

Examples

① $f(x_1 \ldots x_n) = x_1$



② $f(x_1 \ldots x_n) = \mathrm{maj}(x_1 \ldots x_n)$

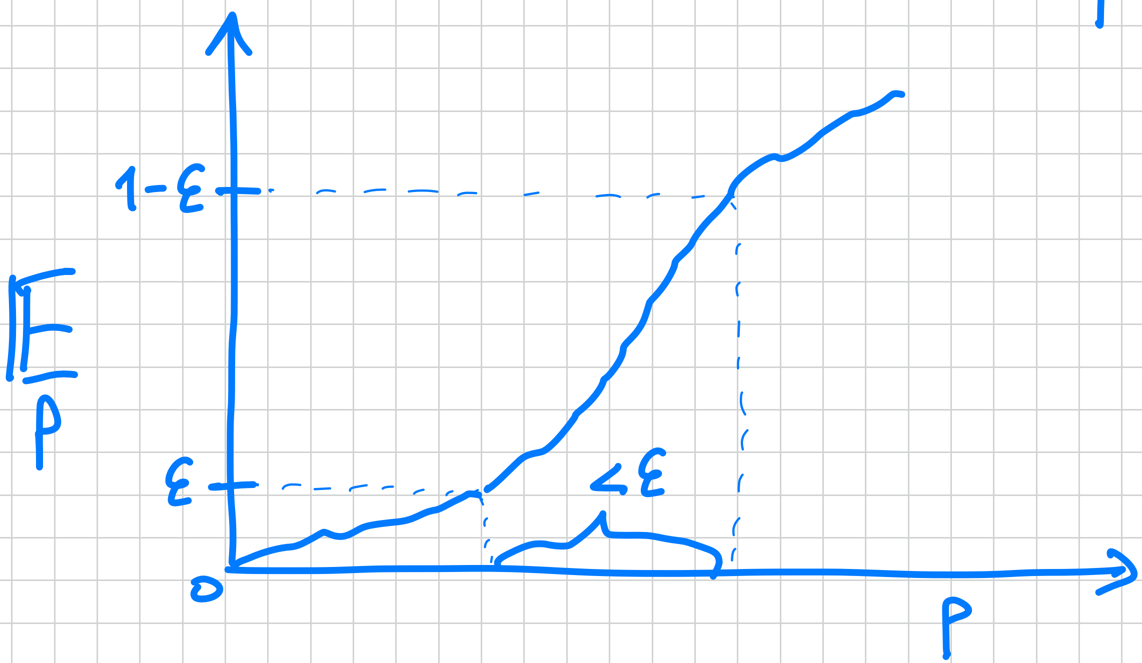


Ordered PCSPs

monotone f: $\quad p \le q \implies \mathbb{E}_p[f] \le \mathbb{E}_q[f]$

Th $\forall$ Ordered $\mathrm{PCSP}(\mathbb{A},\mathbb{B})$ is either

- in P $\iff \forall \varepsilon\ \exists f \in \mathrm{Pol}$ s.t. $\left| p : \varepsilon < \mathbb{E}_p[f] < 1-\varepsilon \right| < \varepsilon$



or

- NP-hard, assuming Rich 2-to-1 Conjecture

Ordered PCSPs

monotone f: $\quad p \leq q \Rightarrow \mathbb{E}_p[f] \leq \mathbb{E}_q[f]$

Tractability

If $\forall \varepsilon \, \exists f \in \mathrm{Pol}$ s.t. $|p : \varepsilon < \mathbb{E}_p[f] < 1-\varepsilon| < \varepsilon$

then MAX/MIN/THR_t

Proof

random minor is symmetric with $\Pr > 0$

Ordered PCSPs

monotone f: $\quad p \leq q \Rightarrow \mathbb{E}_p[f] \leq \mathbb{E}_q[f]$

Hardness

If $\exists \varepsilon \, \forall f \in \mathrm{Pol} \; \left| p : \varepsilon < \mathbb{E}_p[f] < 1 - \varepsilon \right| > \varepsilon$

then Rich 2-to-1 hard $\quad$ ($\exists$ Sel s.t. ....)

Proof $\quad$ Fix $f : \{0,1\}^n \to \{0,1\}$

$$\exists p \text{ s.t. } \frac{d}{dp} \mathbb{E}_p[f] < \frac{1}{\varepsilon}$$

$$\frac{d}{dp} \mathbb{E}_p[f] = \mathbb{E}_p\left[\#\{i \mid f(\bar{x}) \neq f(\bar{x} \oplus i)\}\right]$$

average sensitivity

Ordered PCSPs

Proof (hardness)

$$\exists p \text{ s.t. } \mathop{\mathbb{E}}_{p}\left[\#\{i \mid f(\bar{x}) \neq f(\bar{x} \oplus i)\}\right] < \frac{1}{\varepsilon}$$

average sensitivity

Basics

Fix $p \in [0,1]$ and $f : \{0,1\}^n \to \{0,1\}$

Let $\forall i \quad x_i = \begin{cases} 1 & \text{with } \Pr = p \\ 0 & \text{with } \Pr = 1-p \end{cases}$

$$\mathrm{Inf}_j^p(f) := \Pr_{x_i}\left[f(\bar{x}) \neq f(\bar{x} \oplus j)\right]$$

Examples

1. $f(x_1 \dots x_n) = x_1$

   $\mathrm{Inf}_1 = 1$

   $\mathrm{Inf}_2 = 0$

2. $\min(x_1 \dots x_n)$

   $\mathrm{Inf}_i = p^{n-1}$

Ordered PCSPs

Proof (hardness)

$$\exists p \text{ s.t. } \mathbb{E}_{p}\left[\#\{i \mid f(\bar{x}) \neq f(\bar{x} \oplus i)\}\right] < \frac{1}{\varepsilon}$$

average sensitivity

Friedgut's Th: then $\exists$ monotone $g : \{0,1\}^n \to \{0,1\}$

— depends on $C = C(\varepsilon)$ coords $J \subseteq [n]$

— $\Pr_{p}[f(\bar{x}) \neq g(\bar{x})] < \gamma = \gamma(\varepsilon)$

Claim: $\forall q \; \exists j \in J \quad \mathrm{Inf}_j^q(f) > \tau(\varepsilon)$

so $\exists j \in J \quad |q : \mathrm{Inf}_j^q(f) > \tau(\varepsilon)| > \frac{1}{C}$

Ordered PCSPs

Proof

$$\exists j \in J \quad |q : \mathrm{Inf}_j^q(f) > \tau(\varepsilon)| > \frac{1}{c}$$

original

$$\varphi_j(f) = \int_0^1 \mathrm{Inf}_j^p(f)\,dp > \frac{\tau}{C}$$

BGS'21: $\forall i \quad \varphi_i(f) > \Delta$

$$\Downarrow$$

$$\mathop{\mathbb{E}}_{\pi}\left[\varphi_{\pi(i)}(f^{\pi})\right] > \Delta'$$

2-to-1

new

We: $\forall i \quad \mathrm{Inf}_i^p(f) > \Delta$

$$\Downarrow$$

$$\mathop{\mathbb{E}}_{\pi}\left[\mathrm{Inf}_{\pi(i)}^p(f^{\pi})\right] > \Delta'$$

2-to-1

Influence preservation

beyond monotone

Lemma

$\forall f$ s.t. $\sum_i \mathcal{I}nf_i^p(f) < \gamma n$

$$\forall i \quad \mathcal{I}nf_i^p(f) > \Delta \Rightarrow \mathop{\mathbb{E}}_{\substack{\pi \\ 2\text{-to-}1}}\left[\mathcal{I}nf^p_{\pi(i)}(f^\pi)\right] > \Delta'$$

Consequences

f is unate if $\forall_i \left(\forall \overline{x} \quad f(x_1 \ldots 0 \ldots x_n) \leq f(x_1 \ldots 1 \ldots x_n)\right)$

or

$\left(\forall \overline{x} \quad f(x_1 \ldots 0 \ldots x_n) \geq f(x_1 \ldots 1 \ldots x_n)\right)$

f is k-PTF if $f(\overline{x}) = \begin{cases} 1 & P(\overline{x}) > 0 \\ 0 & \text{else} \end{cases}$

for some polynomial P of deg $\leq k$

Th If $\forall f \in Pol$ is unate or $\forall f \in Pol$ is k-PTF

and $\exists_{\varepsilon, p} \forall f \quad \max_i Inf_i^p(f) > \varepsilon$

then Rich 2-to-1 hard

Consequences

f is *unate* if $\forall_i \left(\forall \overline{x} \quad f(x_1 \dots 0 \dots x_n) \leq f(x_1 \dots 1 \dots x_n) \right)$

or

$\left(\forall \overline{x} \quad f(x_1 \dots 0 \dots x_n) \geq f(x_1 \dots 1 \dots x_n) \right)$

$$\mathbb{E}_{p,q}[f] := \mathbb{E}_{x_i}\left[f(x_1 \dots x_n)\right] \quad \text{where} \quad x_i = \begin{cases} 1 & \text{with Pr } p \text{ or } q \\ 0 & \end{cases}$$

Tractability

assume $\forall f \in \mathrm{Pol}$ is unate

$\nexists f \quad \forall \varepsilon \; \exists f \quad$ s.t. $\left| (p,q) : \varepsilon < \mathbb{E}_{p,q}[f] < 1 - \varepsilon \right| < \varepsilon$

then $MAX/MIN/THR_t/AT$

Open questions

- Dichotomy for unate/k-PTF PCSPs?
- Capturing existing results