Towards infinite PCSP: a dichotomy for monochromatic cliques

Demian Banakh & Alexey Barsukov & Tamio-Vesa Nakajima

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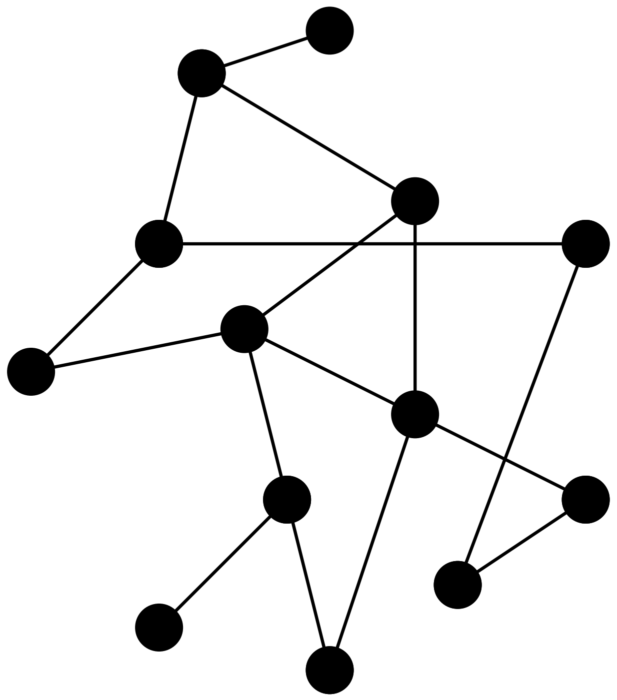
Coloring graphs avoiding monochromatic triangles



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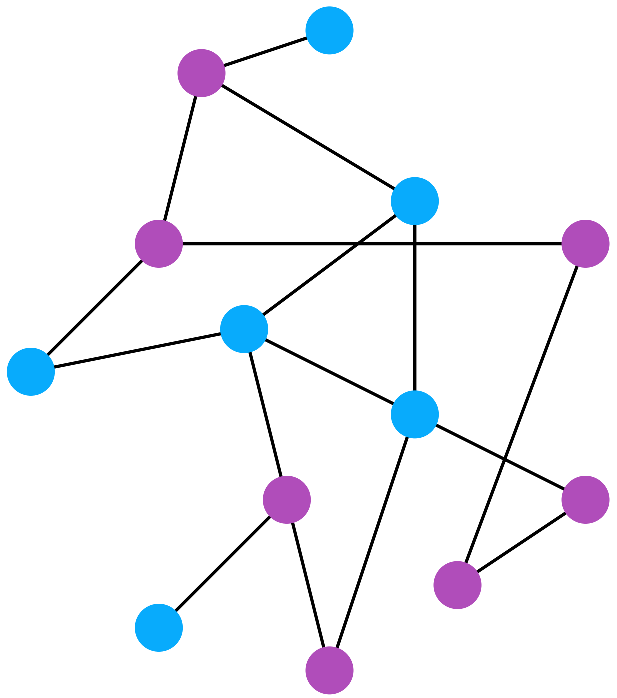

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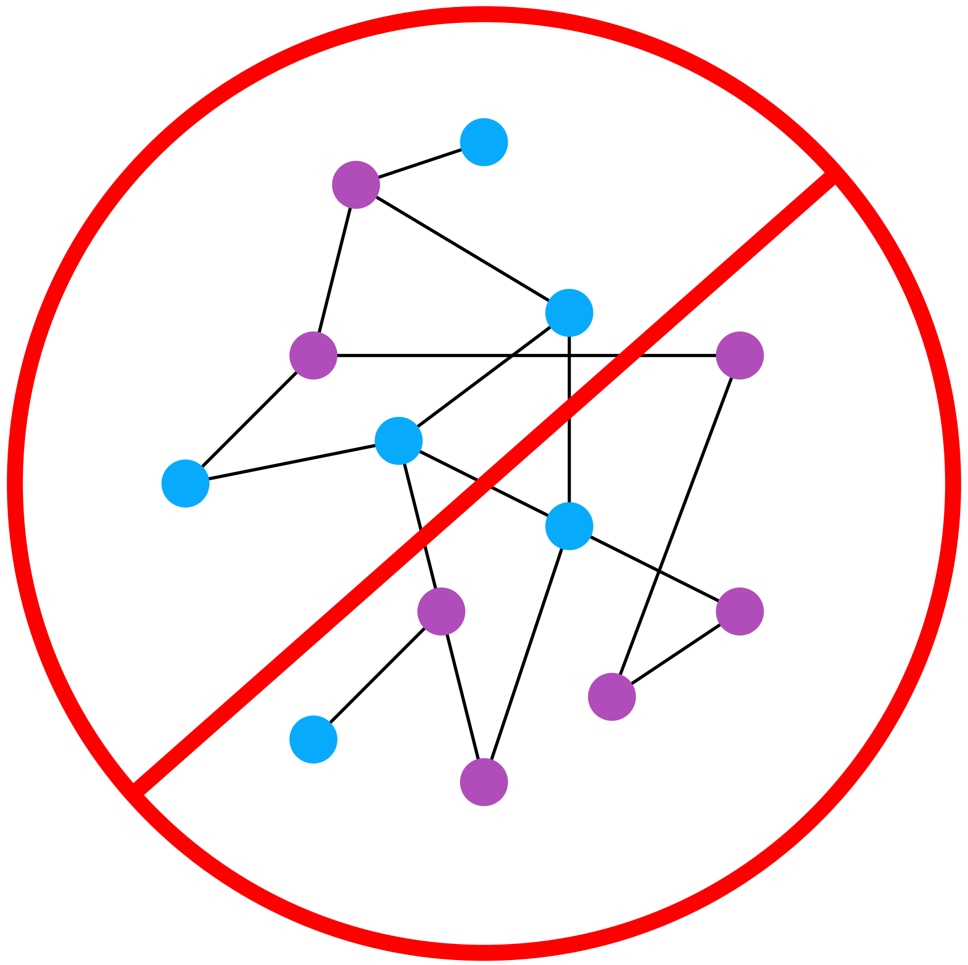

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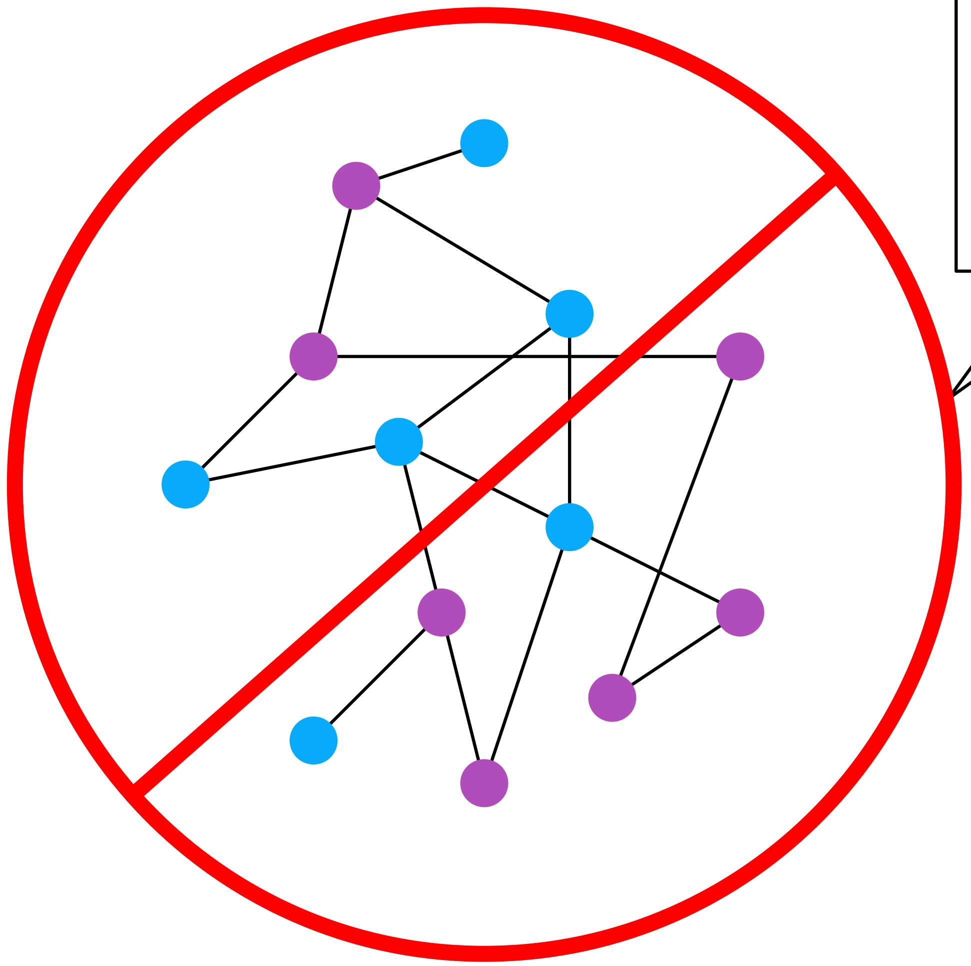

Forbidden patterns

find a coloring that is free of

Coloring graphs avoiding monochromatic triangles

MMSNP logic

[FV'98]

$\exists A, B \forall x, y, z:$

$A(x) \iff \neg B(x)$

$\neg(A(x) \wedge A(y) \wedge A(z) \wedge K_3(x, y, z))$

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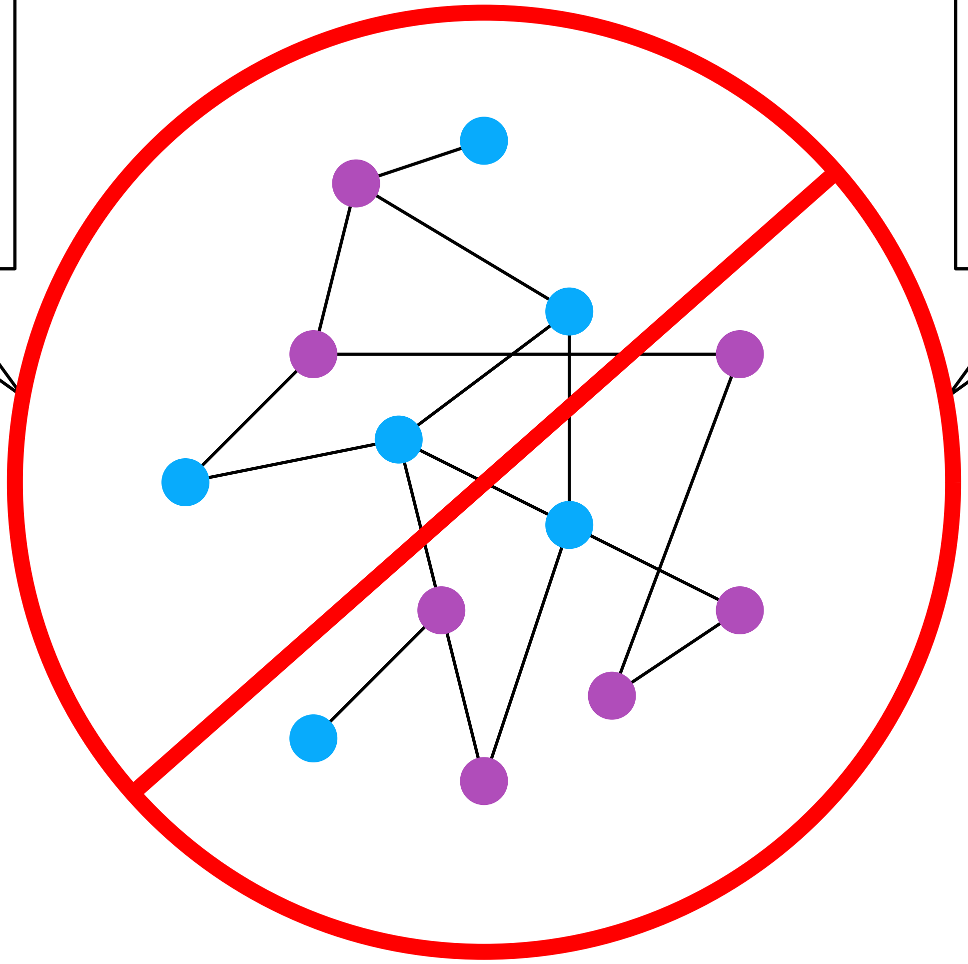

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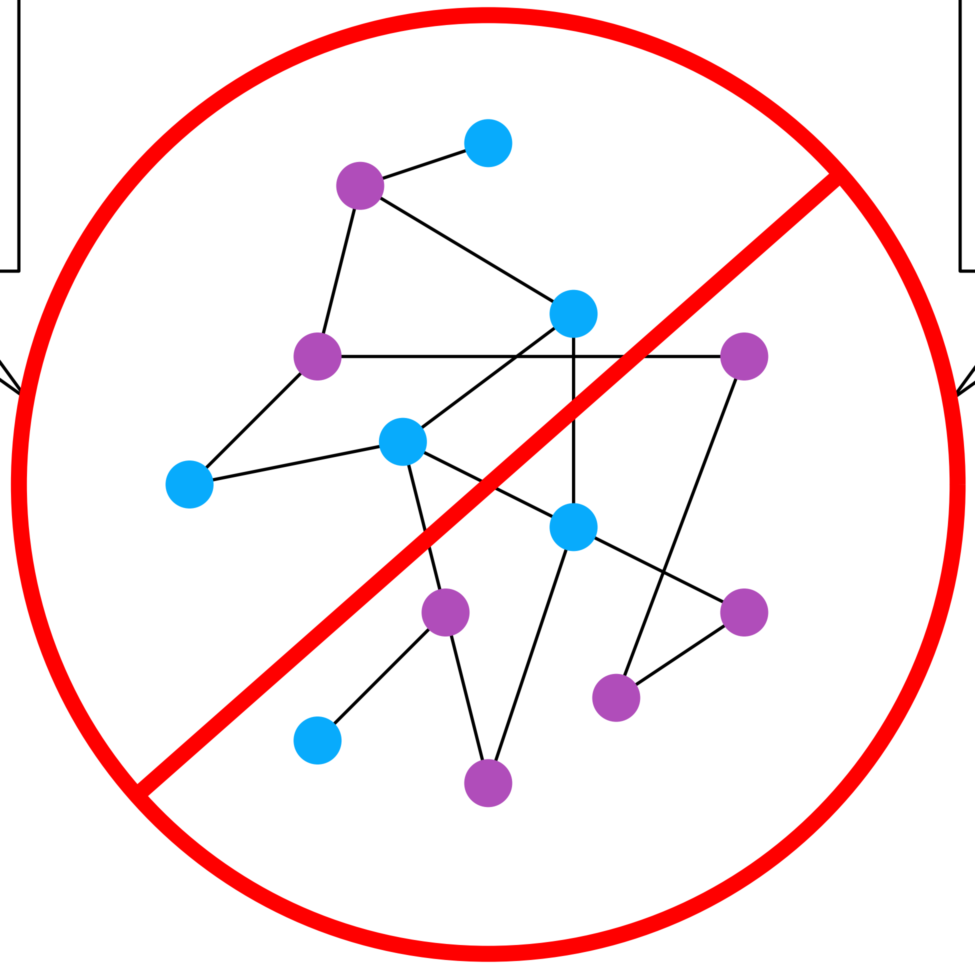

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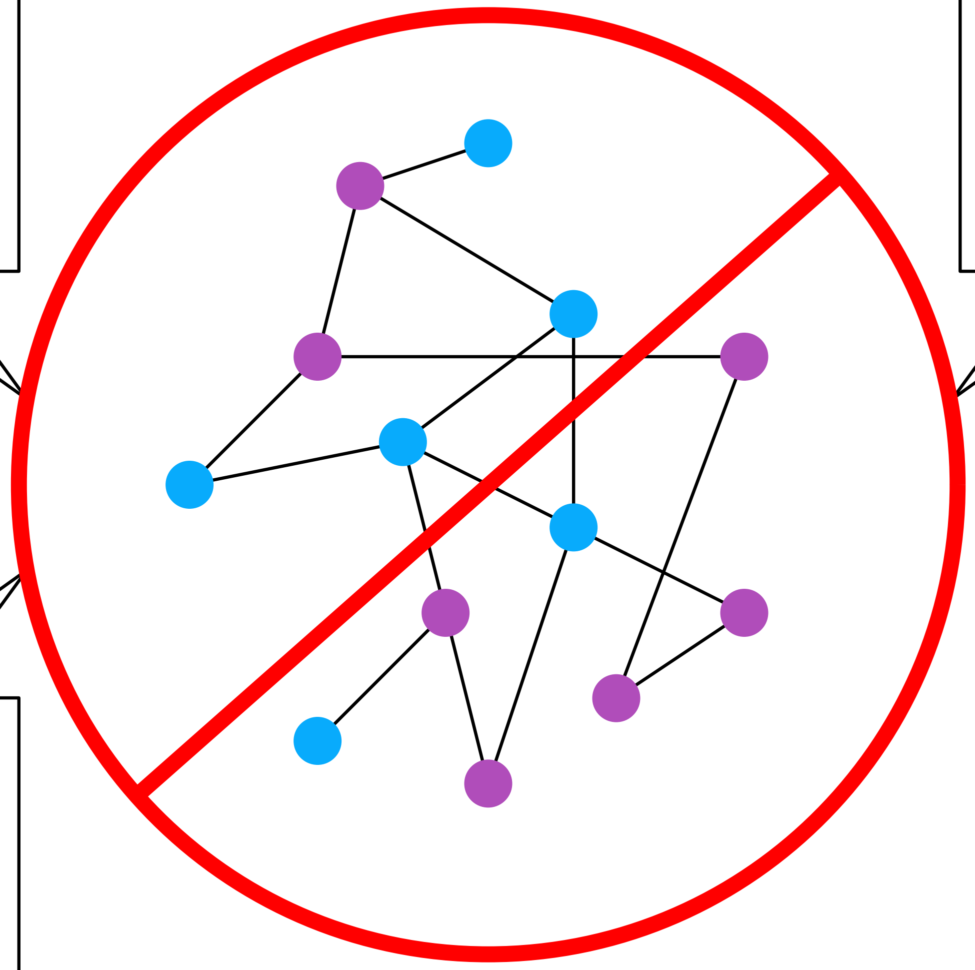
[MS'07]



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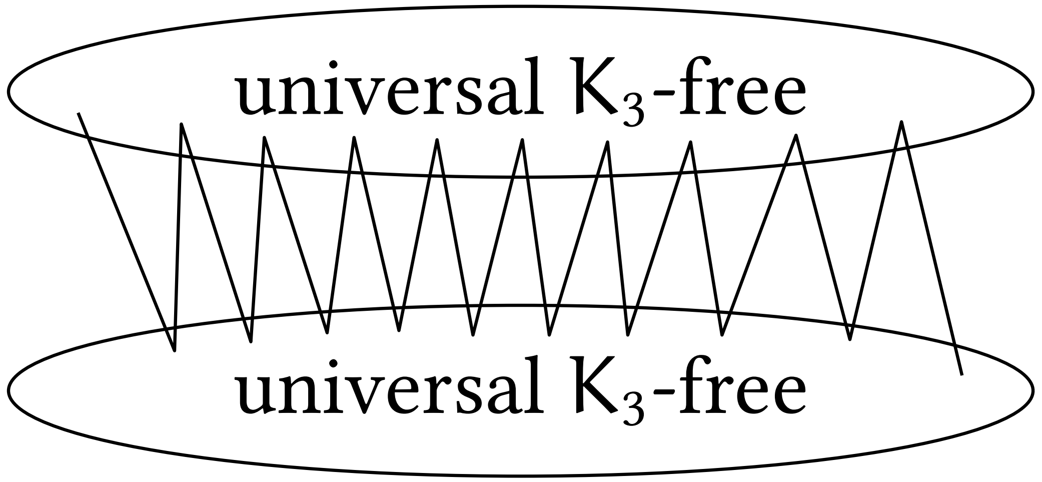
Infinite-domain CSP



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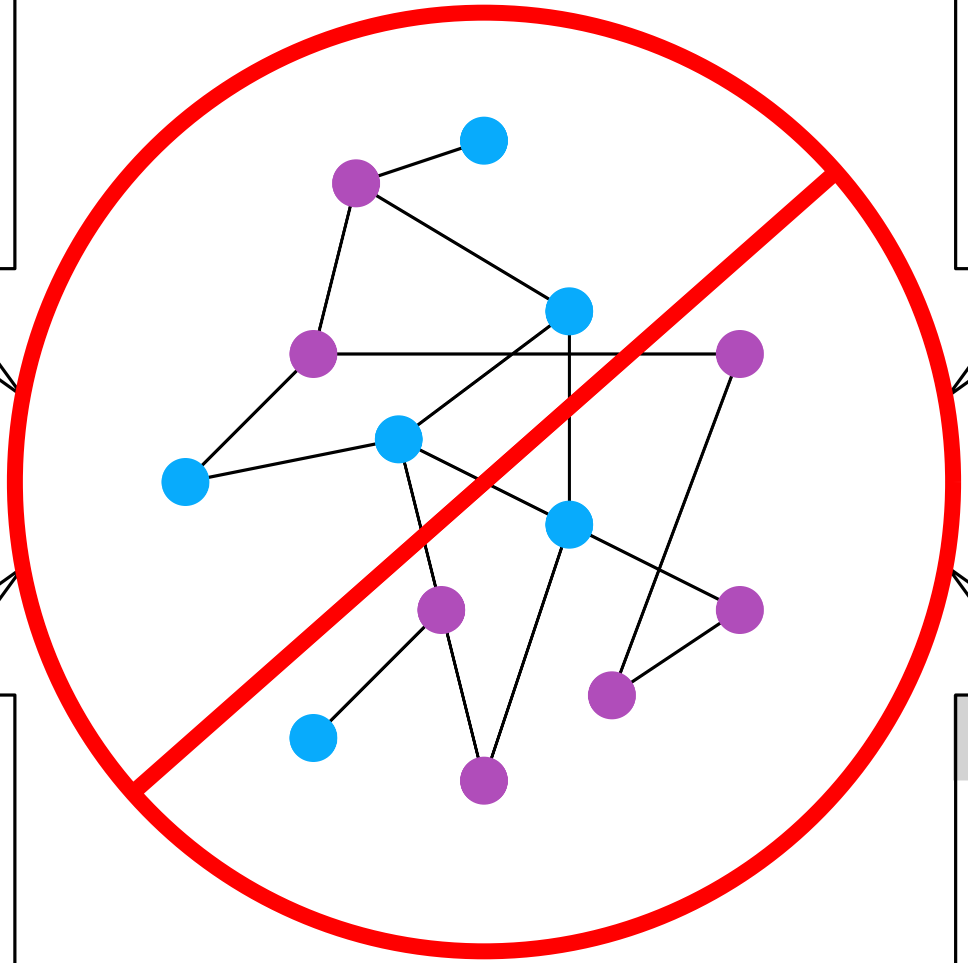

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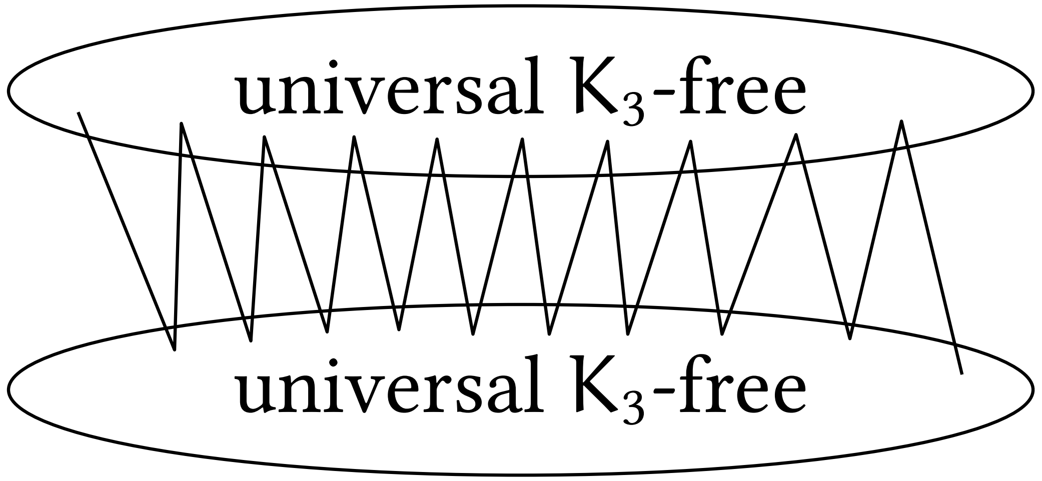

Finite-domain CSP

not expressible

but poly-time equivalent to one

[FV'98, Kun'13]

(finite-domain) CSP

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CSP(R) where $R \subseteq A^r$

IN: r-uniform hypergraph $H \subseteq X^r$

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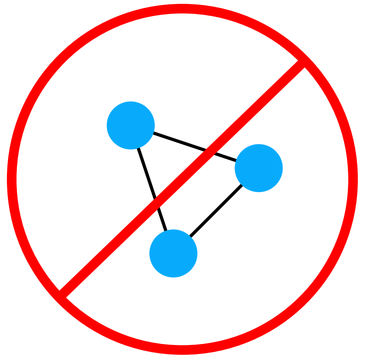

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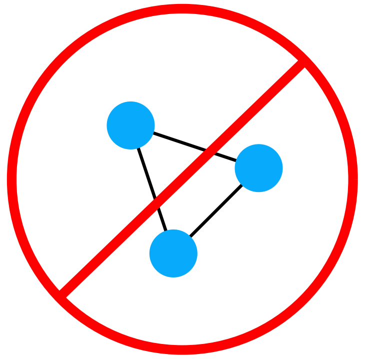

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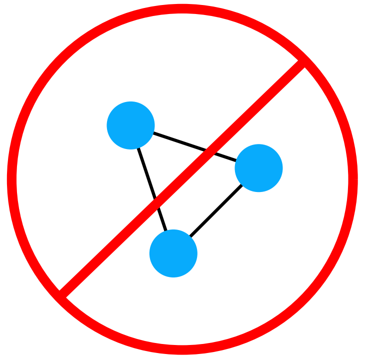

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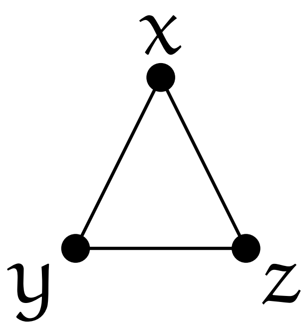
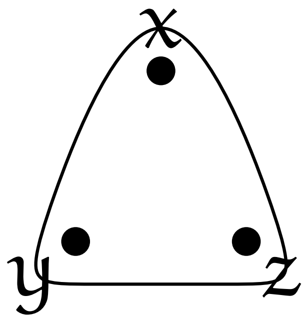


hyperedge



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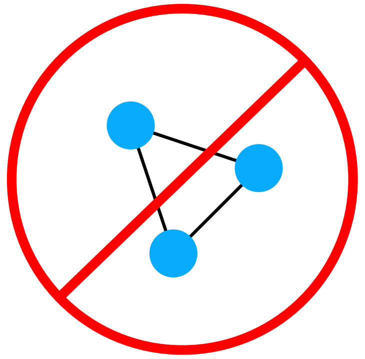

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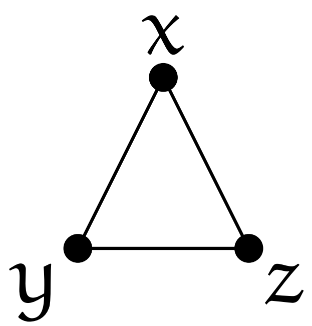

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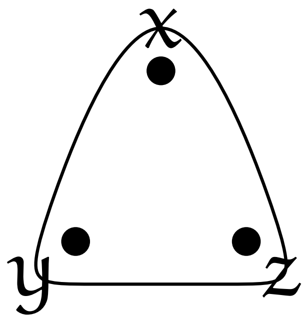

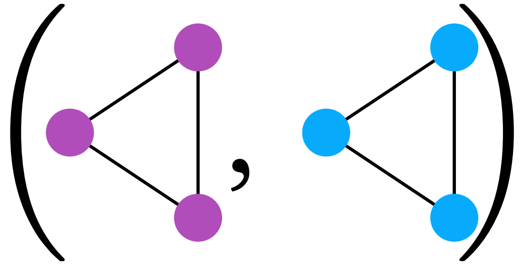

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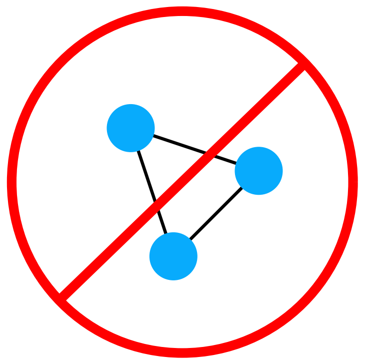

$\equiv_{\log}$

$\mathrm{CSP}(\mathrm{NAE}_3^{(2)})$ NP-complete

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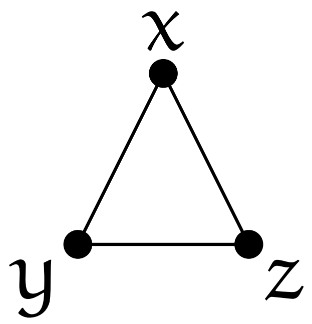
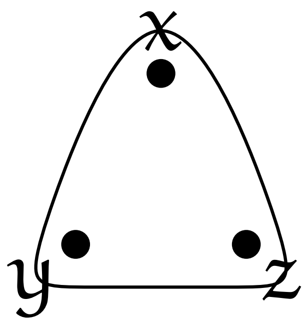
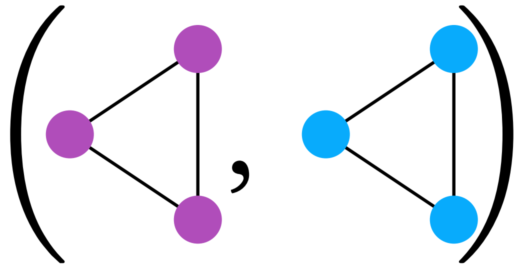


$\longleftrightarrow$

hyperedge

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Promise setting

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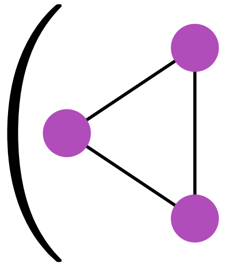
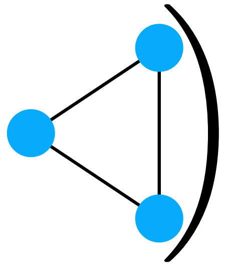
hard to find

Suppose G has (, )-free coloring.

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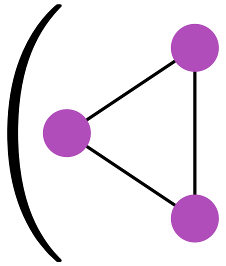
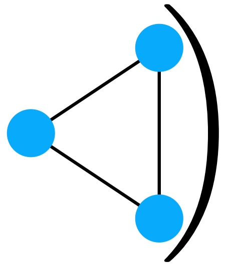
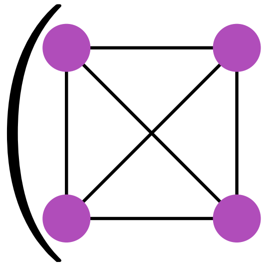
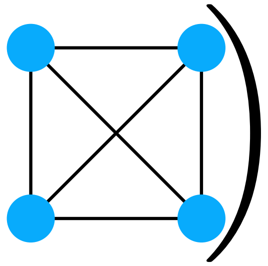
Suppose G has ( , )-free coloring.

Find ( , )-free coloring.

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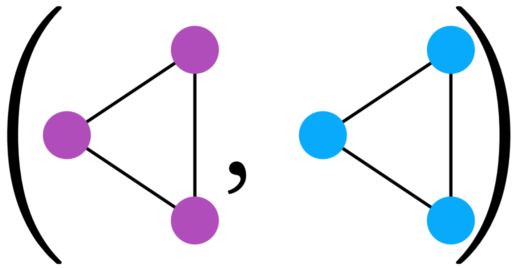
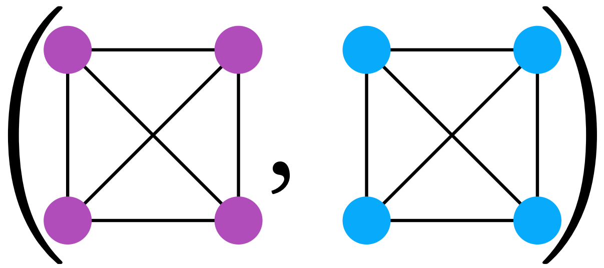
Suppose G has  -free coloring.

Find  -free coloring.

Possible in polynomial time!

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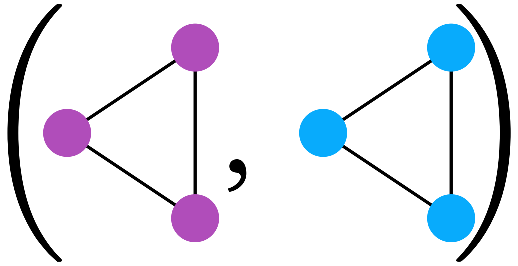
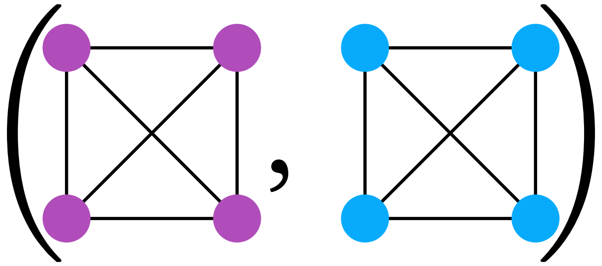
Suppose G has  -free coloring.

Find  -free coloring.

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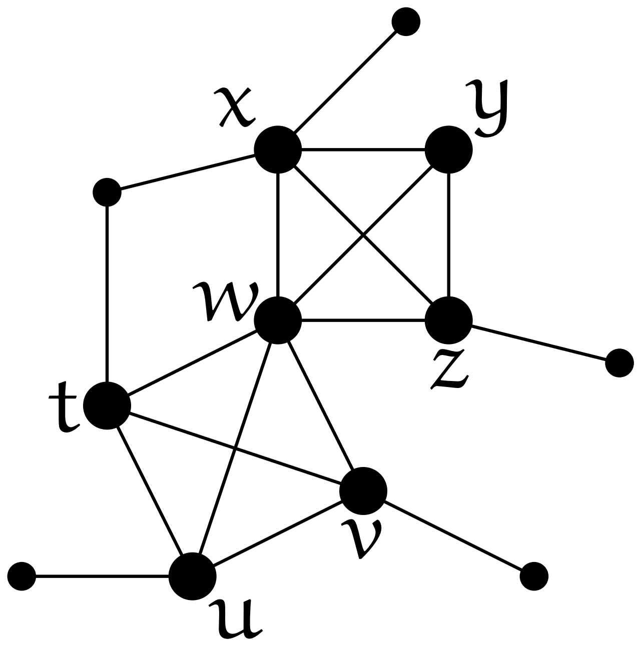

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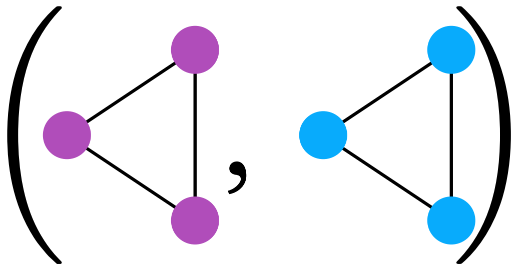
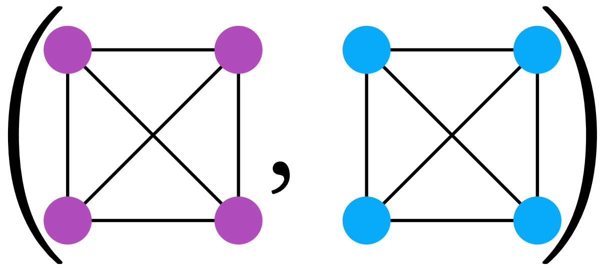
Suppose G has  -free coloring.

Find  -free coloring.

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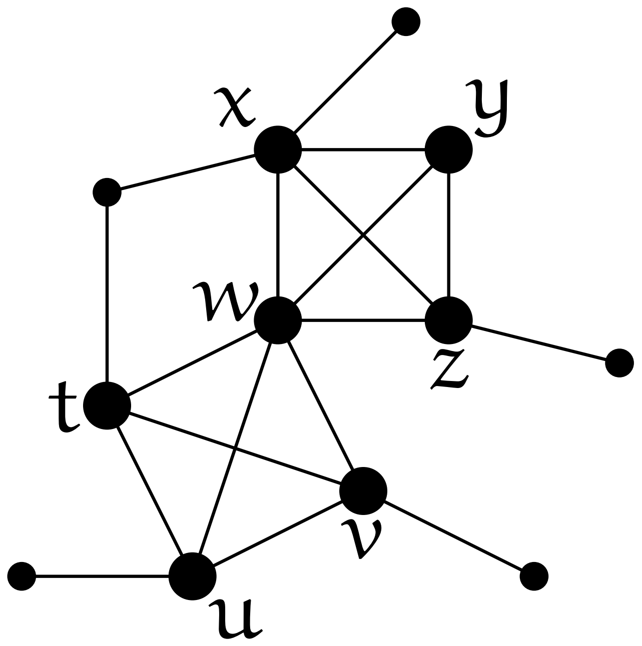

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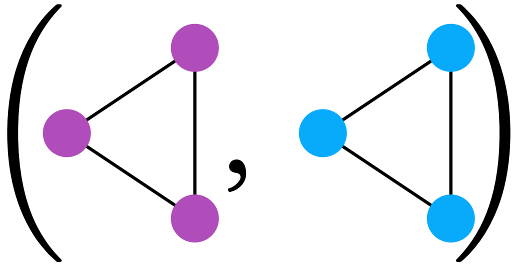
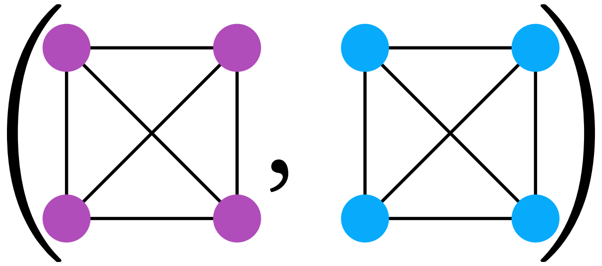
Suppose G has  -free coloring.

Find  -free coloring.

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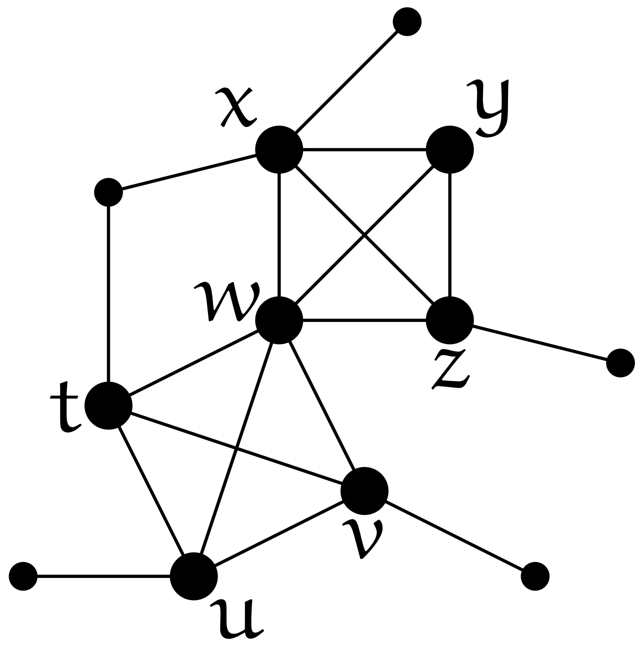

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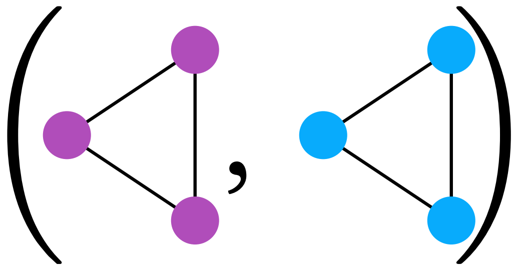
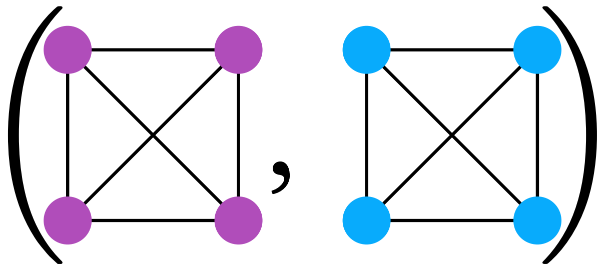
Suppose G has  -free coloring.

Find  -free coloring.

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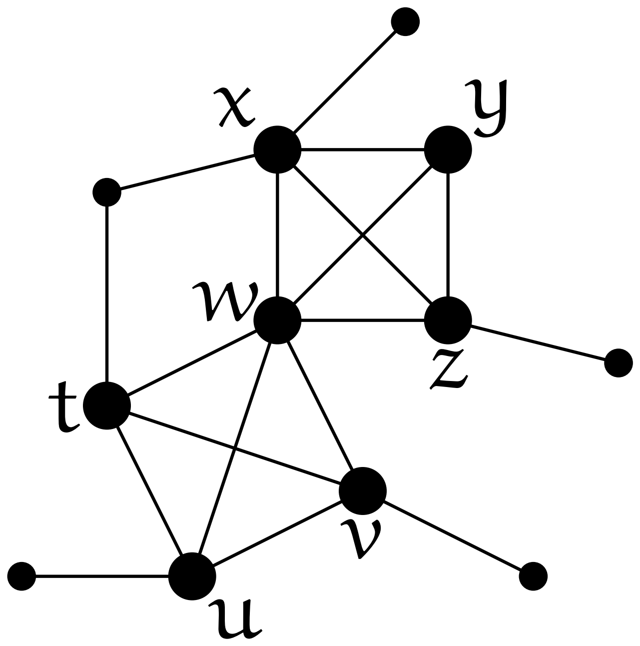

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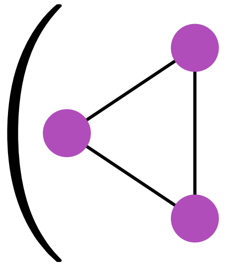
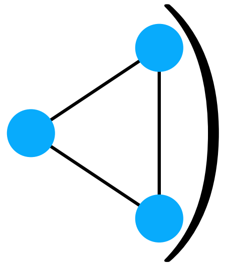
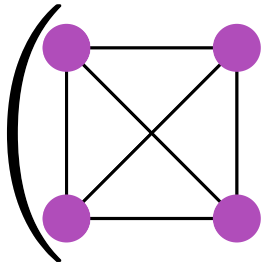
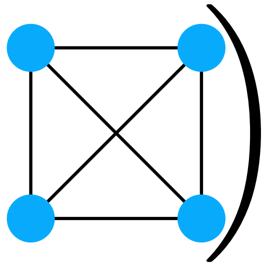
Suppose G has ( , )-free coloring.

Find ( , )-free coloring.

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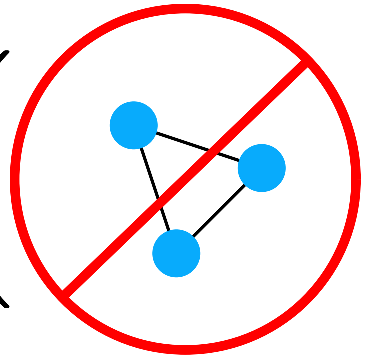
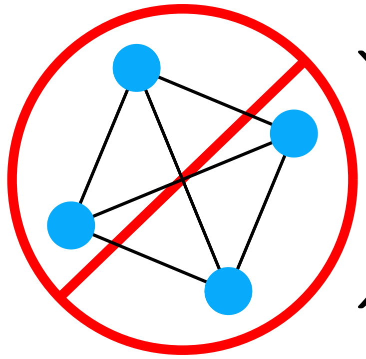
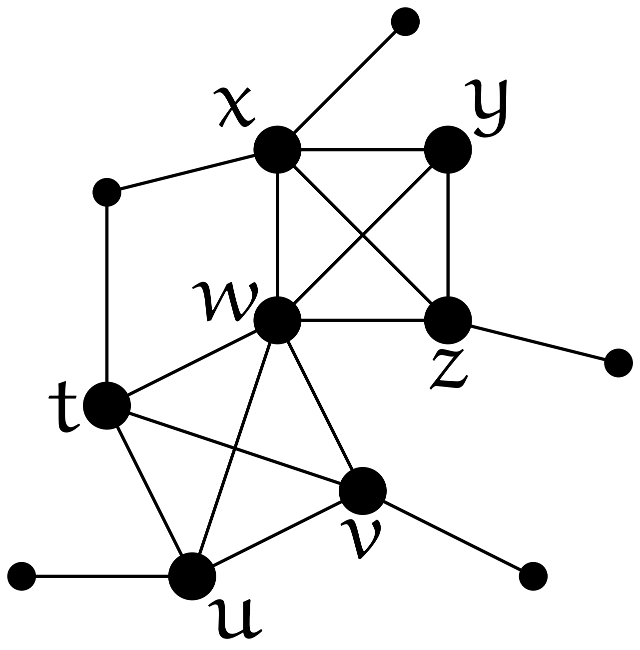
( ; )



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$\mathrm{PCSP}(\{100, 010, 001\};\, \{0,1\}^3 \setminus \{000, 111\})$ also known as $\mathrm{PCSP}(1\mathrm{in}3;\, \mathrm{NAE}_3^{(2)})$

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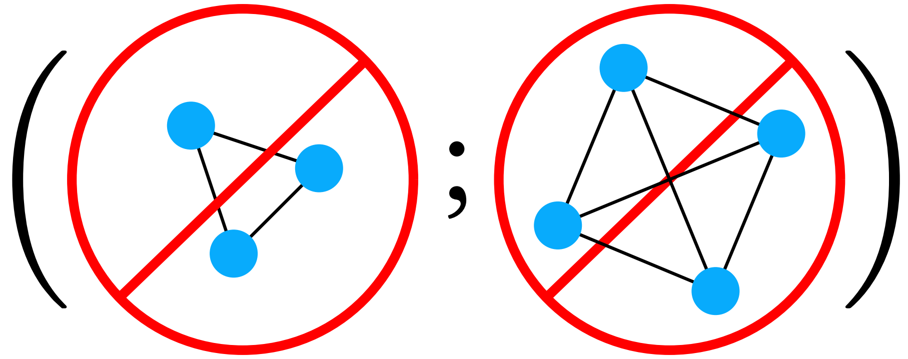

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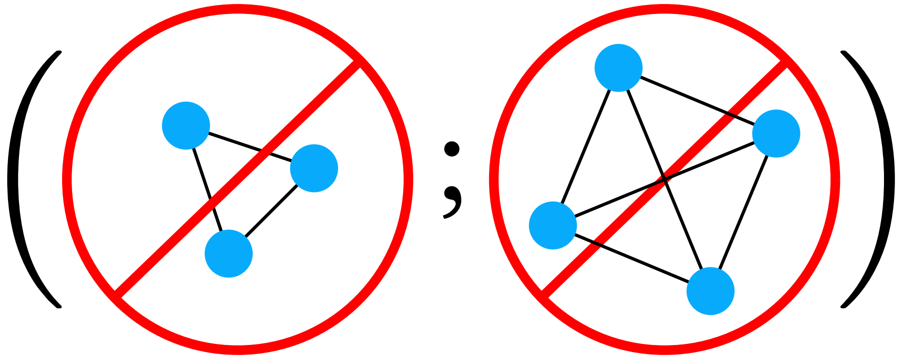

$$\equiv_{\log} \quad \mathrm{PCSP}(2\mathrm{in}4;\ \mathrm{NAE}_4^{(2)})$$

$G = (V; E)$

$H = (V; E')$

(finite-domain) Promise CSP

PCSP(R; S) where $R \subseteq A^r$ and $S \subseteq B^r$
IN: r-uniform hypergraph $H \subseteq X^r$
Promise: $H \to R$
OUT: homomorphism $H \to S$

Examples:
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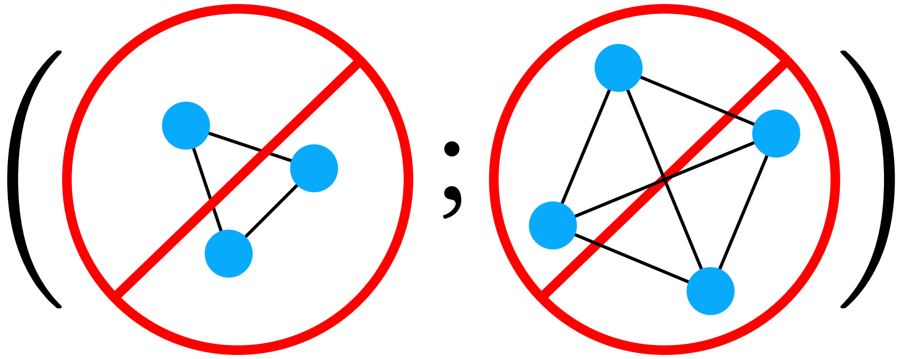
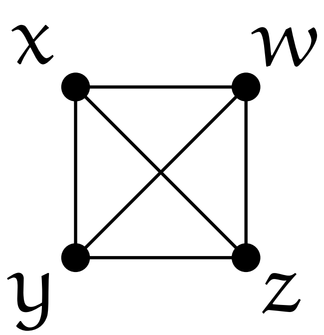


$\equiv_{\log}$ $\mathrm{PCSP}(2\mathrm{in}4; \mathrm{NAE}_4^{(2)})$

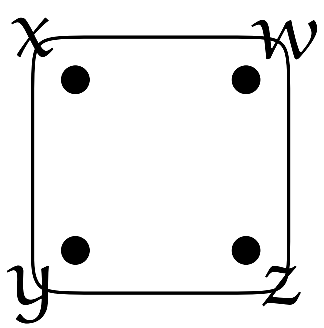
$G = (V; E)$ $H = (V; E')$

4-clique



$\longleftrightarrow$

hyperedge



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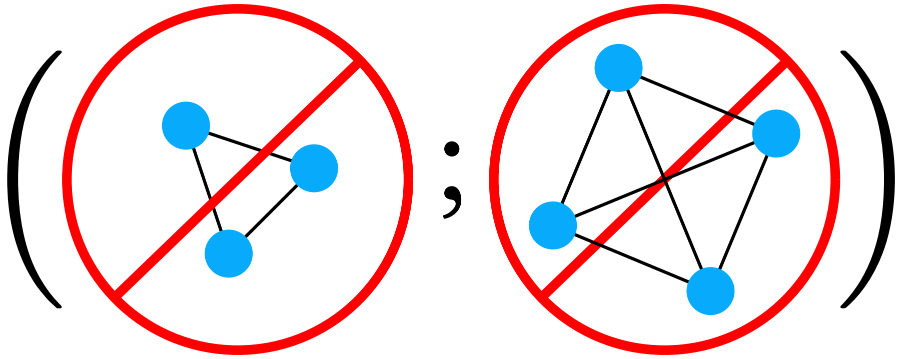

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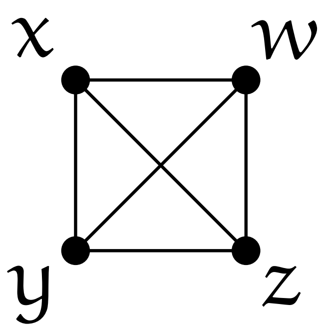

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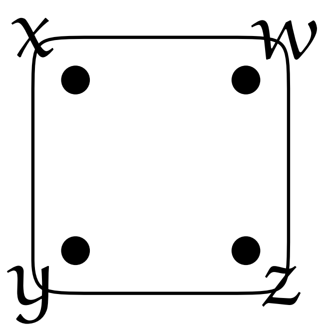

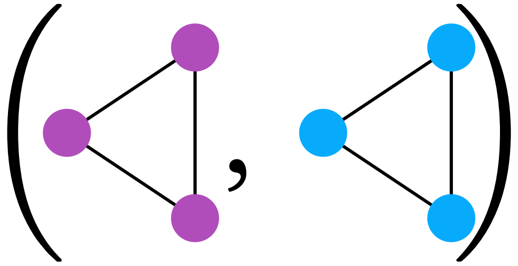

-free coloring of G $=^*$ 2in4-coloring of H

(finite-domain) Promise CSP

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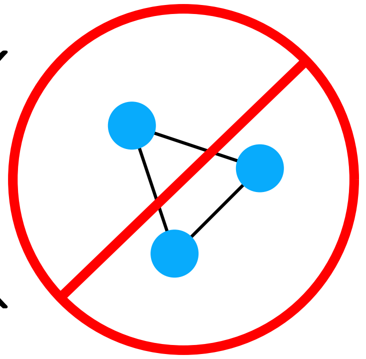
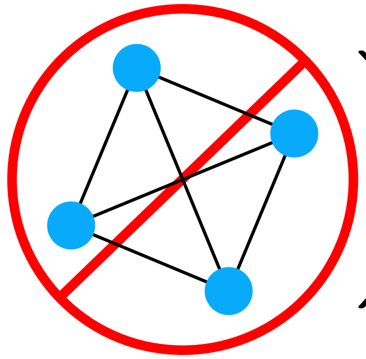
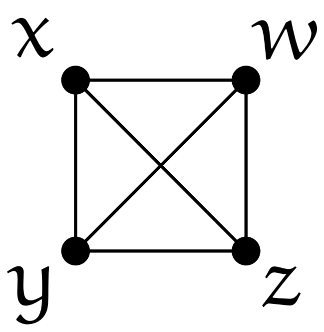
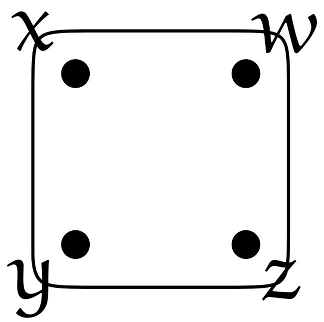
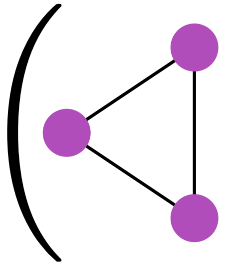
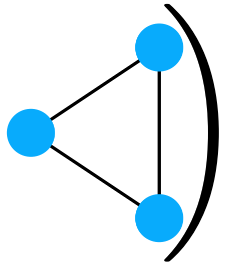
( ; ) $\equiv_{\log}$ $\mathrm{PCSP}(\mathrm{2in4}; \mathrm{NAE}_4^{(2)})$ tractable

$G = (V; E)$ $H = (V; E')$

4-clique 

$\longleftrightarrow$
hyperedge 

( , )-free coloring of G $\equiv^*$ 2in4-coloring of H

More generally

of colors

$$\left(\cancel{K_k^{(c)}} ; \cancel{K_\ell^{(d)}} \right)$$

More generally

of colors

$$\left(\cancel{K_k^{(c)}}\,;\ \cancel{K_\ell^{(d)}}\right) \quad \equiv_{\log} \quad \mathrm{PCSP}\left(R^{(c)}_{<k:\ell};\ R^{(d)}_{<\ell:\ell}\right)$$

More generally

\# of colors

$$\left\{ (a_1, \dots, a_\ell) \in [c]^\ell \;\middle|\; \forall a \ \#\{i : a_i = a\} < k \right\}$$

$$\|$$

$$\left(\cancel{K_k^{(c)}} \,;\, \cancel{K_\ell^{(d)}} \right) \equiv_{\log} \mathrm{PCSP}\left(R^{(c)}_{<k:\ell};\, R^{(d)}_{<\ell:\ell}\right)$$

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$$||$$

$\mathrm{NAE}_\ell^{(d)}$

More generally

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$$\{(a_1,\ldots,a_\ell) \in [c]^\ell \mid \forall a\ \#\{i : a_i = a\} < k\} = R^{(c)}_{<k:\ell}$$

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More generally

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$K_\ell^{(d)}$-free coloring of G $=^*$ $R^{(d)}_{<\ell:\ell}$-coloring of H

Promise coloring graphs avoiding monochromatic cliques

$$\left(\cancel{K_k^{(c)}}\,;\,\cancel{K_\ell^{(d)}}\right)$$

Promise coloring graphs avoiding monochromatic cliques

MMSNP logic

$$\exists A, B \forall x_1, \dots, x_k :$$

$$A(x) \iff \neg B(x)$$

$$\neg(A(x_1) \wedge \cdots \wedge A(x_k) \wedge K_k(x_1, \dots, x_k))$$

$$\neg(B(x_1) \wedge \cdots \wedge B(x_k) \wedge K_k(x_1, \dots, x_k))$$

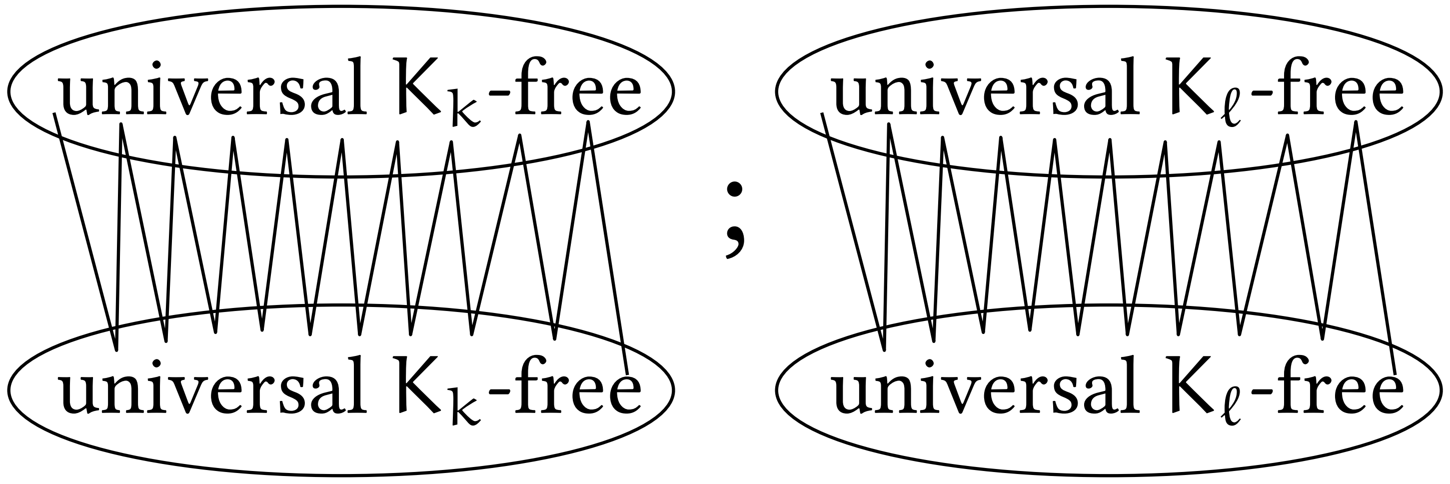
Forbidden patterns

$\exists$ coloring free of

Find a coloring free of

$$\left(\cancel{K_k^{(c)}} \; ; \; \cancel{K_\ell^{(d)}} \right)$$

Infinite-domain PCSP



Finite-domain PCSP

not expressible

but poly-time equivalent to one

$$\mathrm{PCSP}(R^{(c)}_{<k:\ell};\ R^{(d)}_{<\ell:\ell})$$

Classification

$$\left(\cancel{K_k^{(c)}}\,;\,\cancel{K_\ell^{(d)}}\right) \quad \equiv_{\log} \quad \mathrm{PCSP}(R^{(c)}_{<k:\ell};\, R^{(d)}_{<\ell:\ell})$$

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- $\ell \geqslant c(k-1) \implies$ tractable

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$$(\underbrace{1,\ldots,1}_{k-1},\underbrace{2,\ldots,2}_{k-1},\ldots,\underbrace{c,\ldots,c}_{k-1},?)$$

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ℓ-uniform H $\dashrightarrow$ $\begin{cases} x_1+\cdots+x_\ell = k-1 \\ \dots \end{cases}$ $\dashrightarrow$ solve over $\mathbb{Z}$ $\dashrightarrow$ $\begin{matrix} x > 0 \\ x \leqslant 0 \end{matrix}$

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Equivalently: $R^{(c)}_{<k:\ell} \longrightarrow (\mathbb{Z};\ x_1+\cdots+x_\ell = k-1) \longrightarrow \mathrm{NAE}^{(2)}_\ell \longrightarrow \mathrm{NAE}^{(d)}_\ell = R^{(d)}_{<\ell:\ell}$

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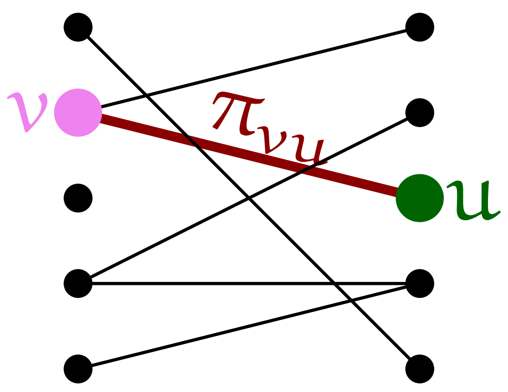
Unique Games



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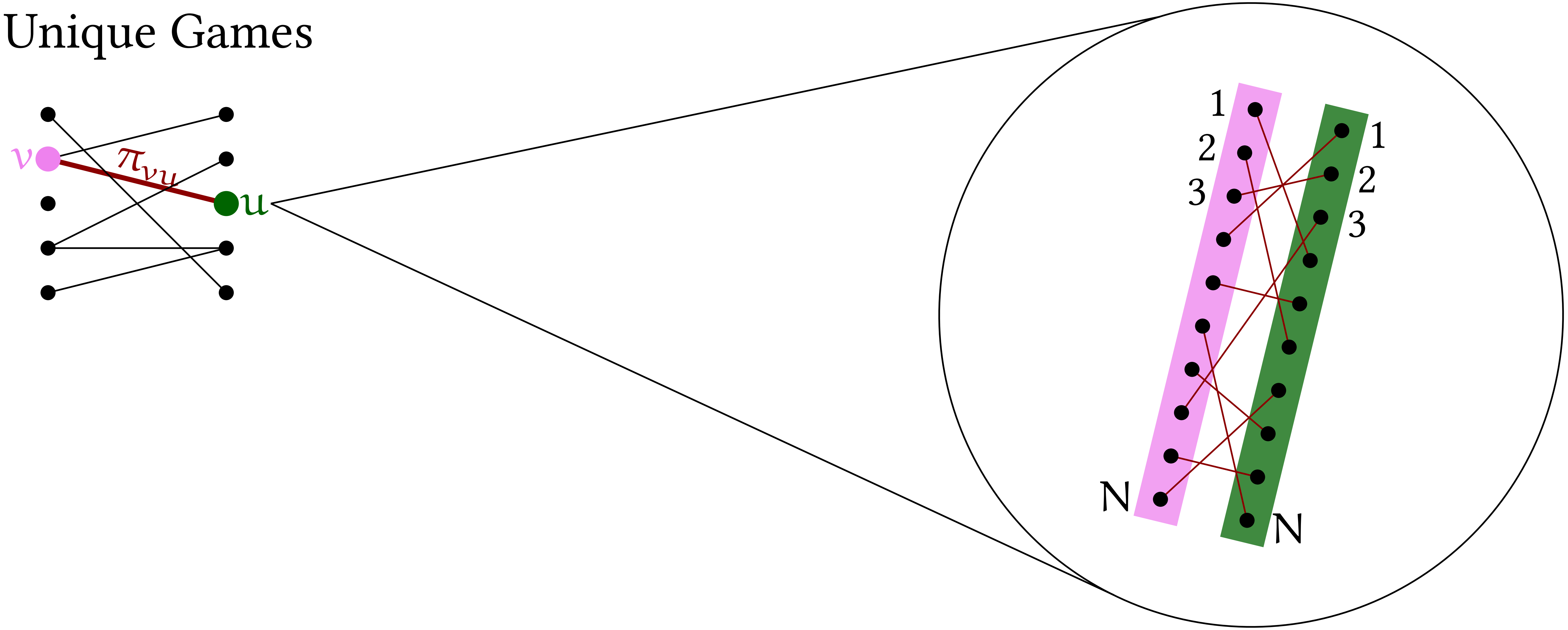
Unique Games



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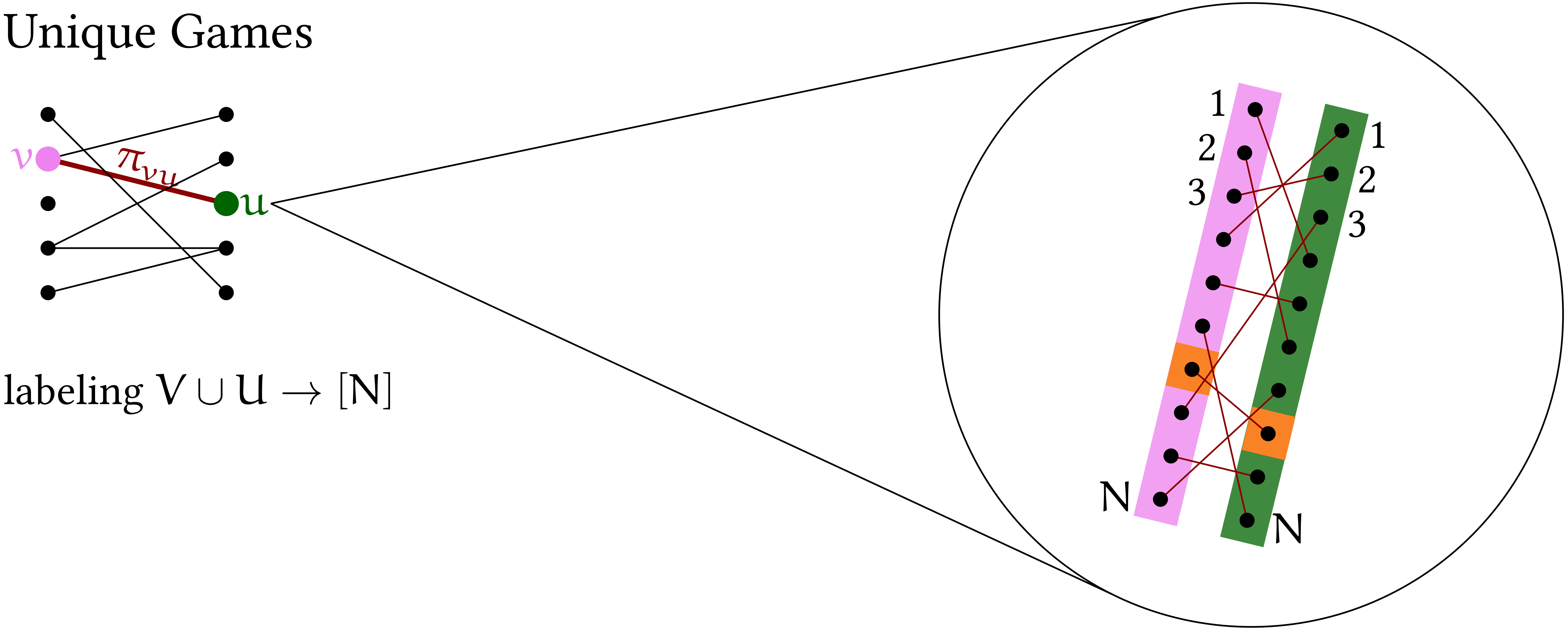
Unique Games



labeling $V \cup U \to [N]$

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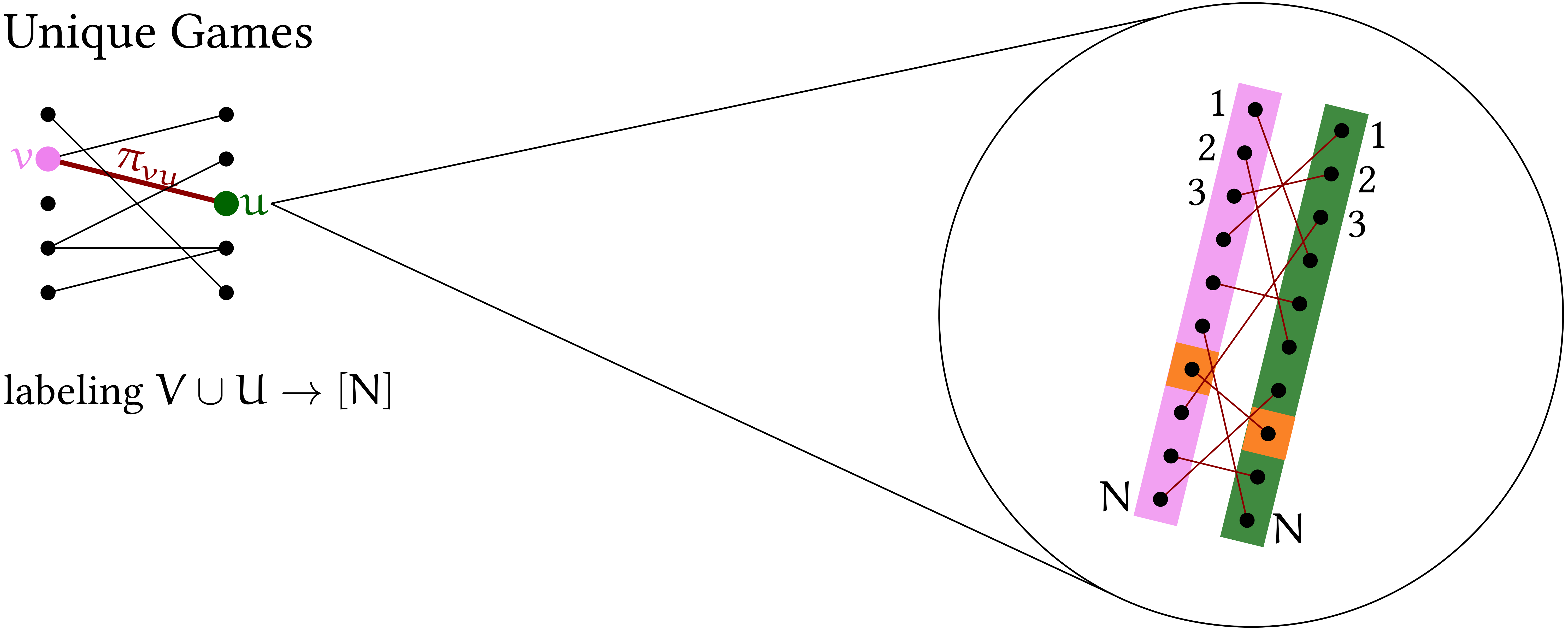
Unique Games



labeling $V \cup U \to [N]$

Conjecture: [Khot'02]

NP-hard to approximately solve unsatisfiable instances

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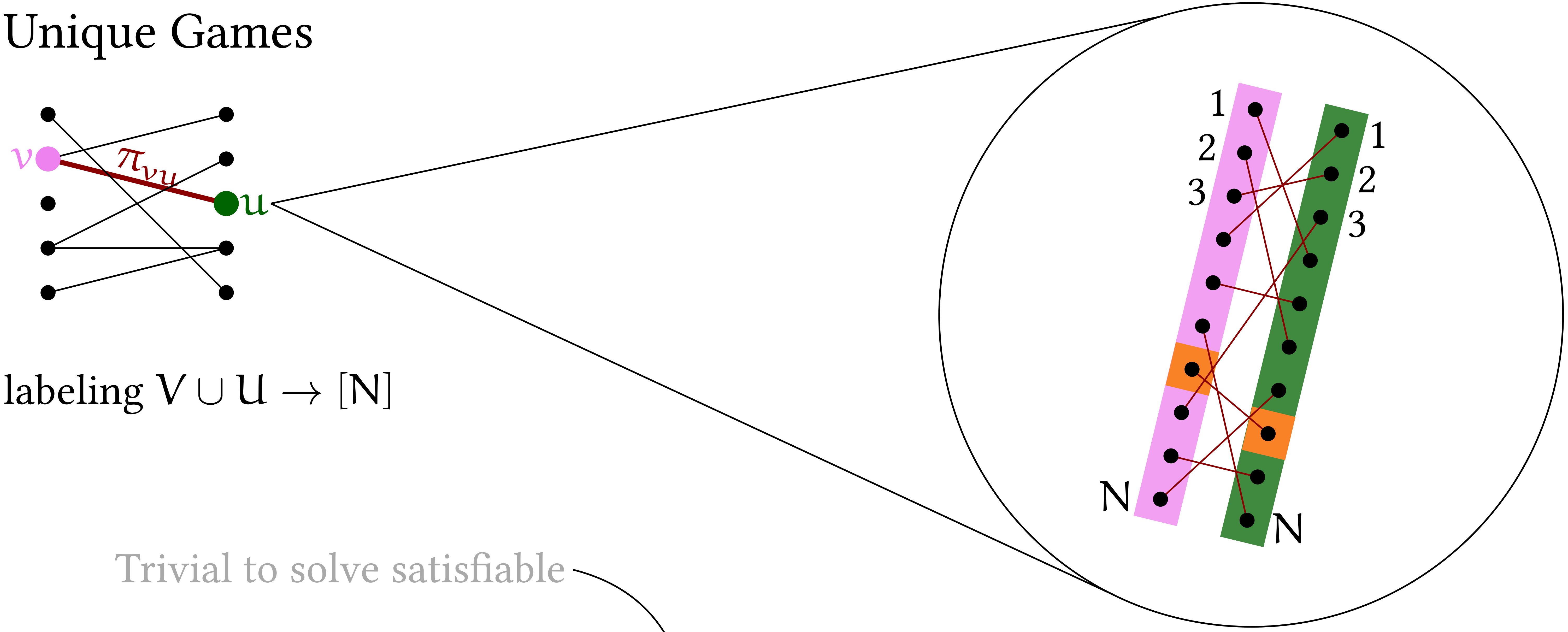
Unique Games



labeling $V \cup U \to [N]$

Trivial to solve satisfiable

Conjecture: [Khot'02]
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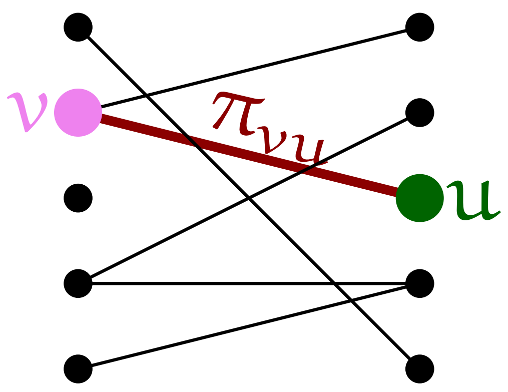
2-to-1 Games



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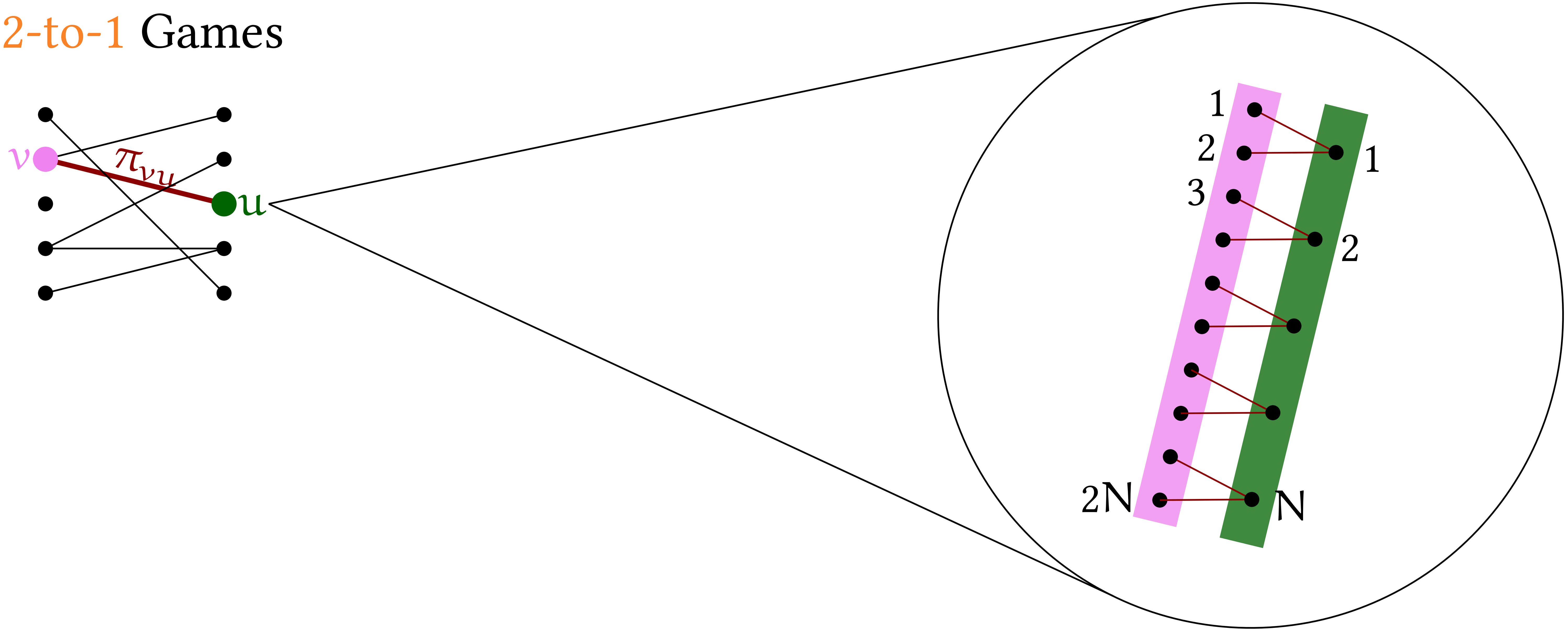
2-to-1 Games



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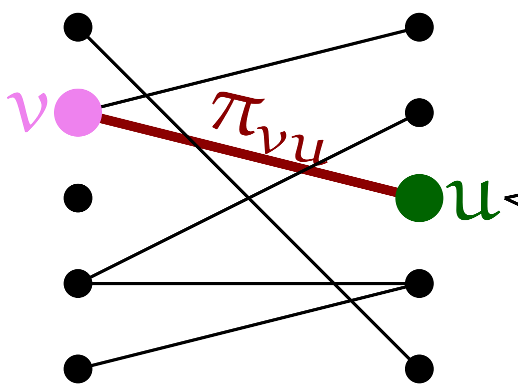
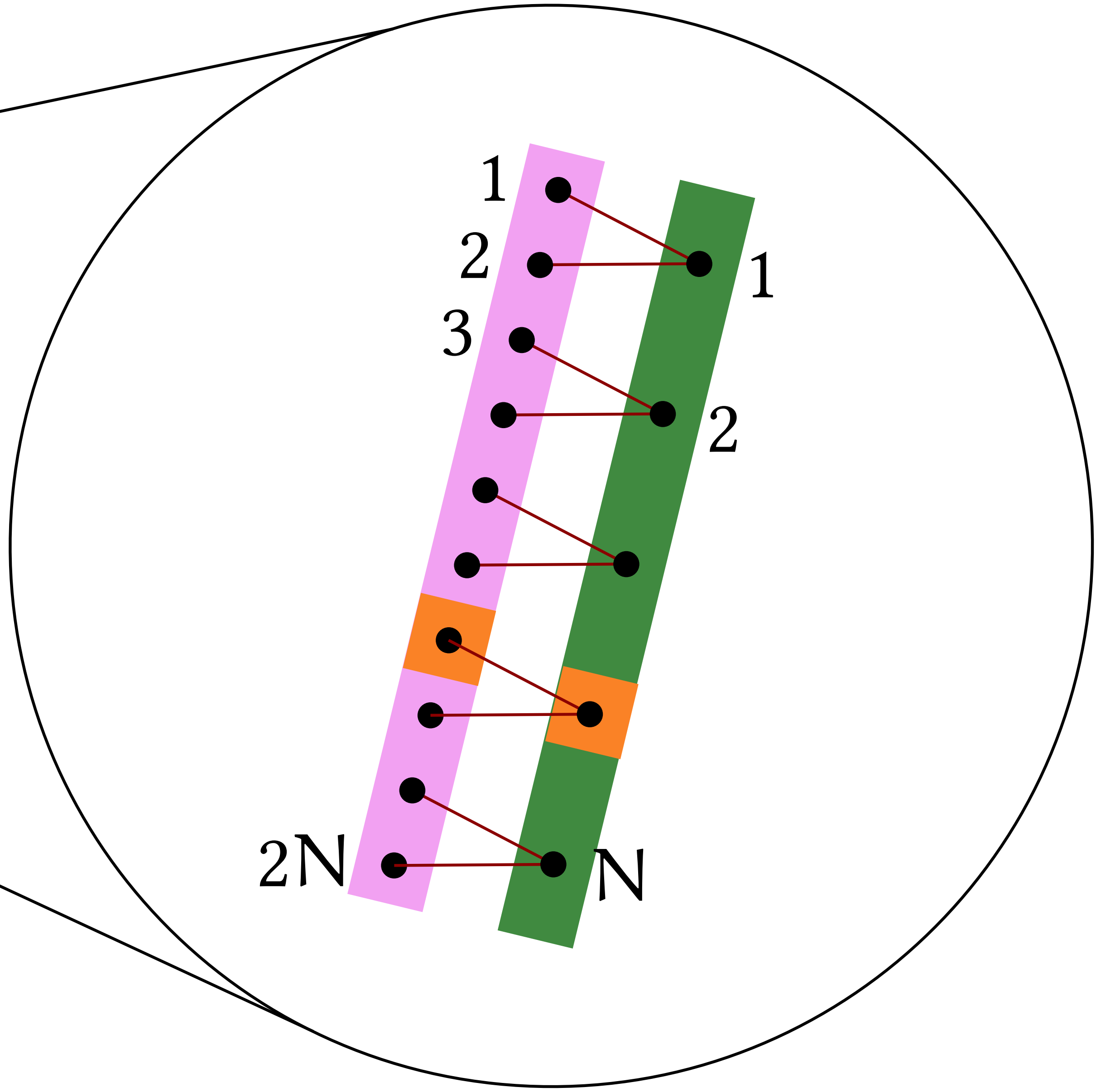
2-to-1 Games





labeling $(V \to [2N]) \cup (U \to [N])$

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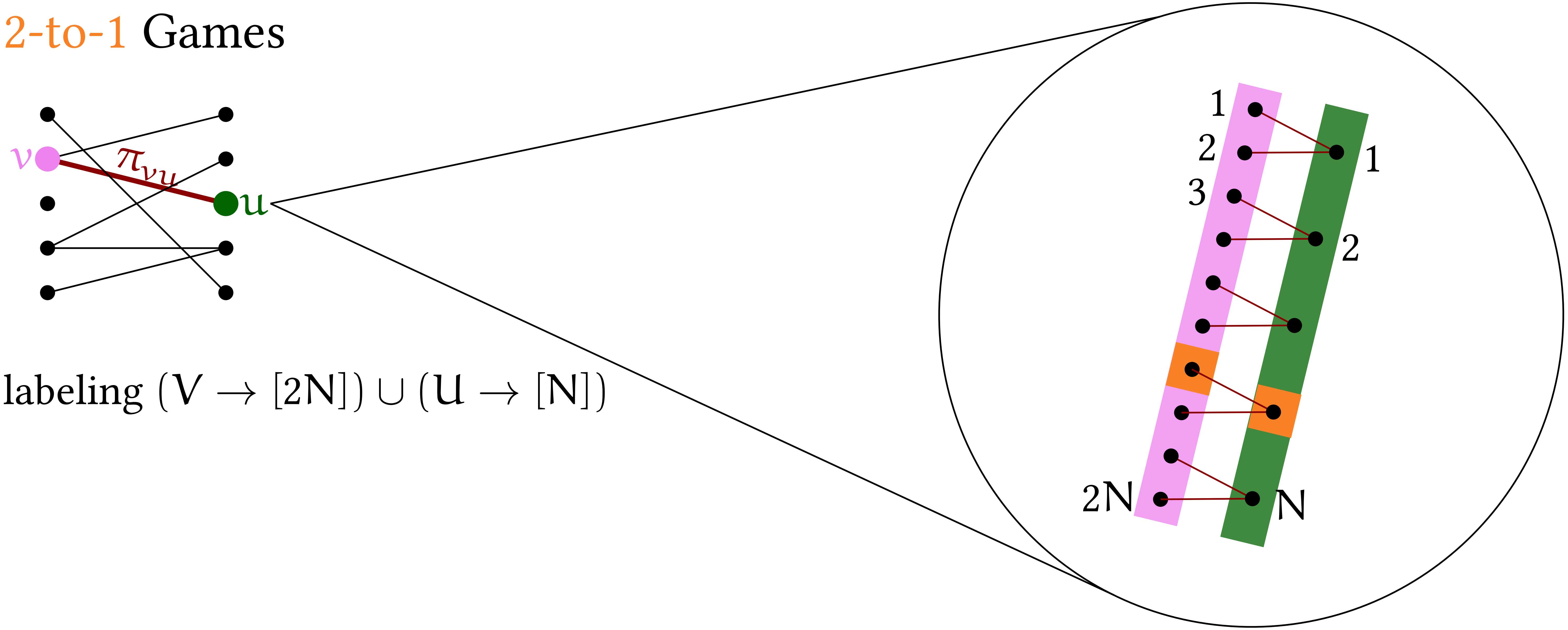
2-to-1 Games



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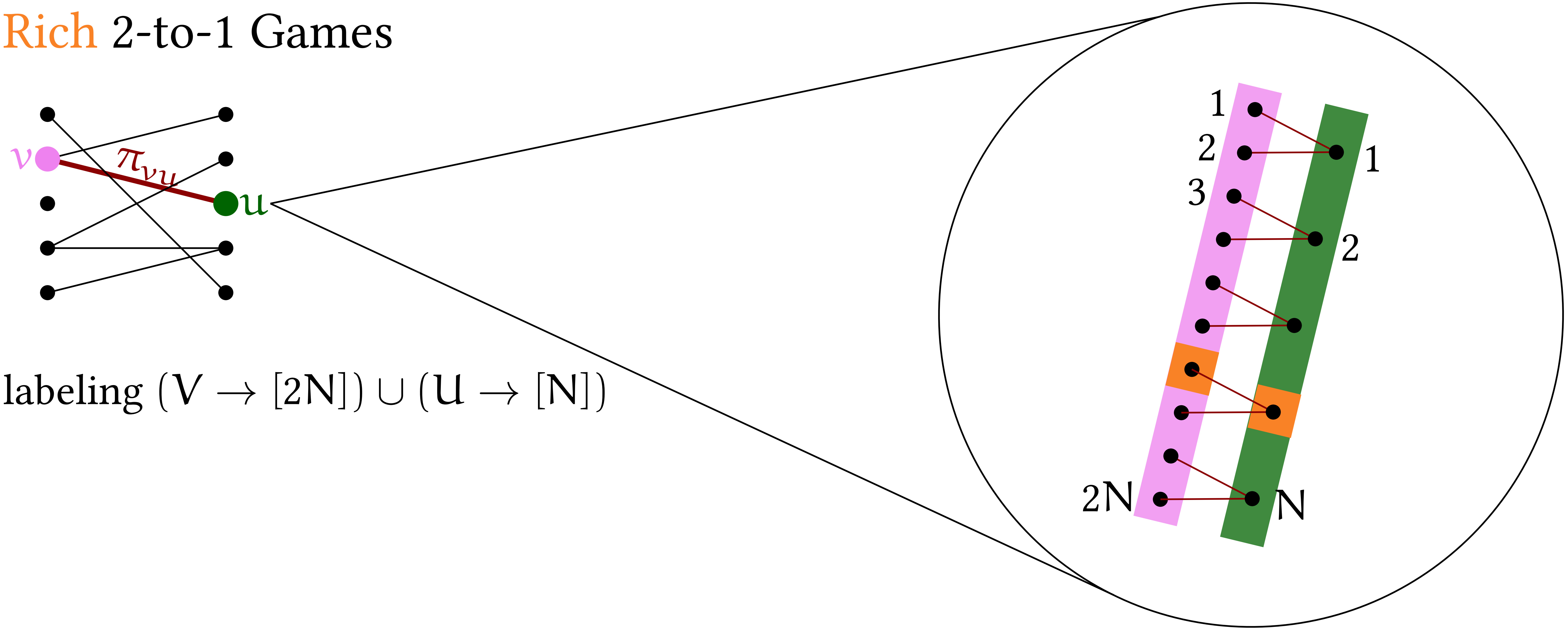
Rich 2-to-1 Games



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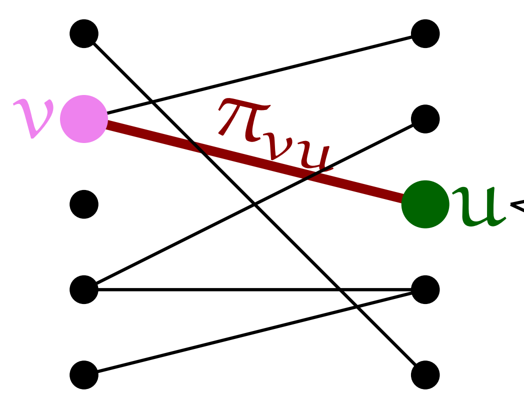
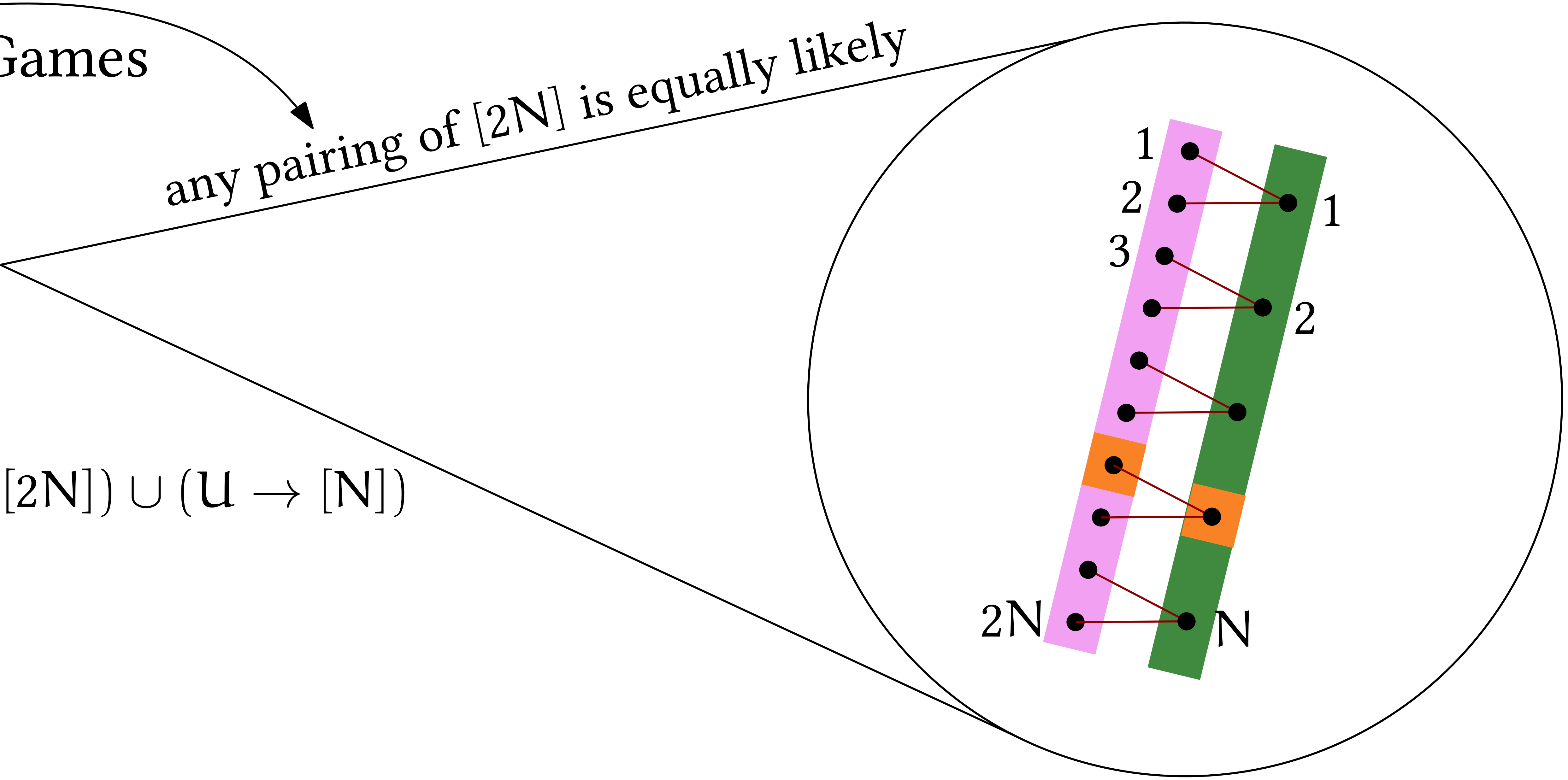
Rich 2-to-1 Games





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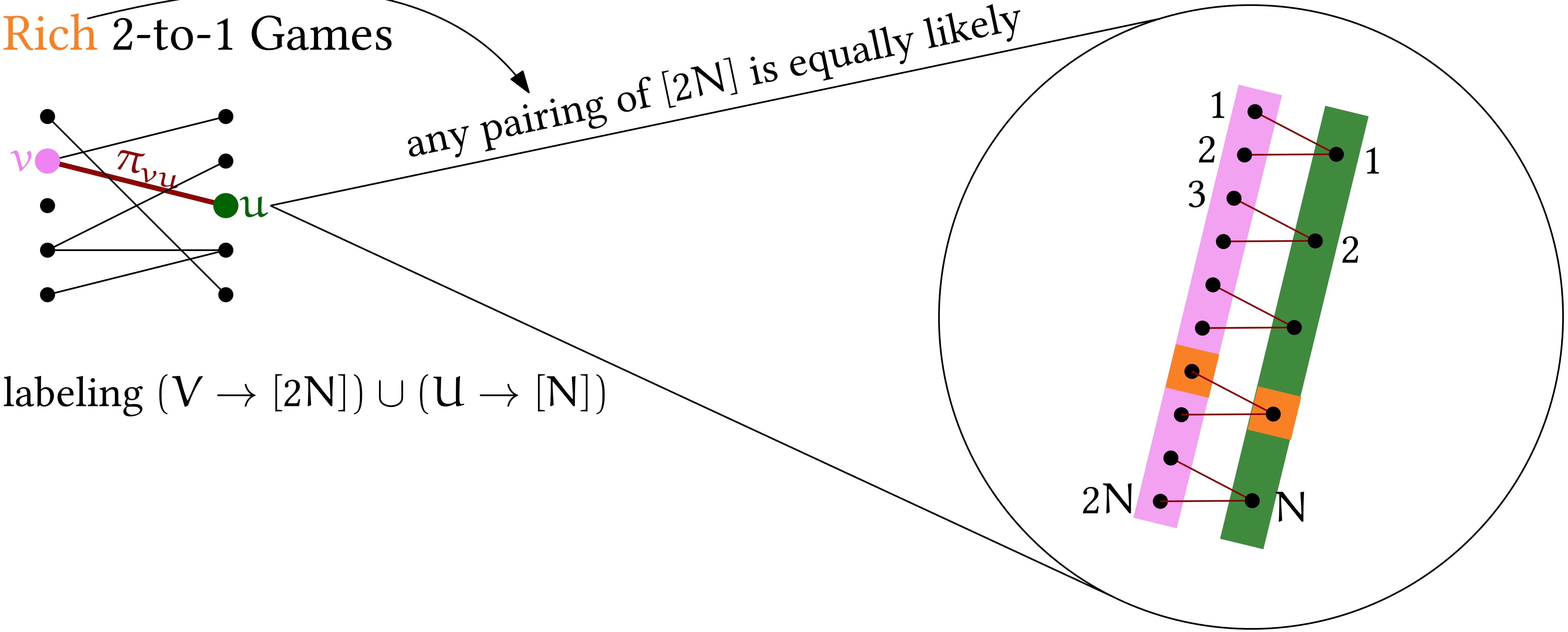
Rich 2-to-1 Games



labeling $(V \to [2N]) \cup (U \to [N])$

Conjecture: [BKM'21]

NP-hard to approximately solve any instances

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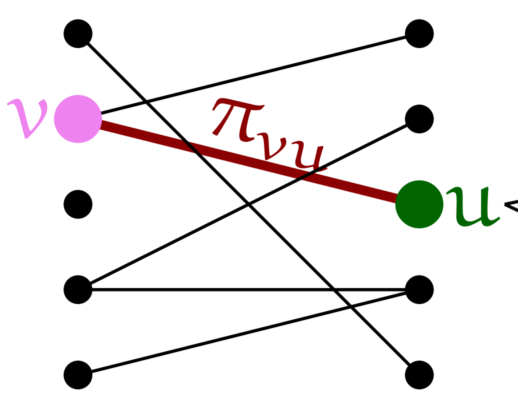
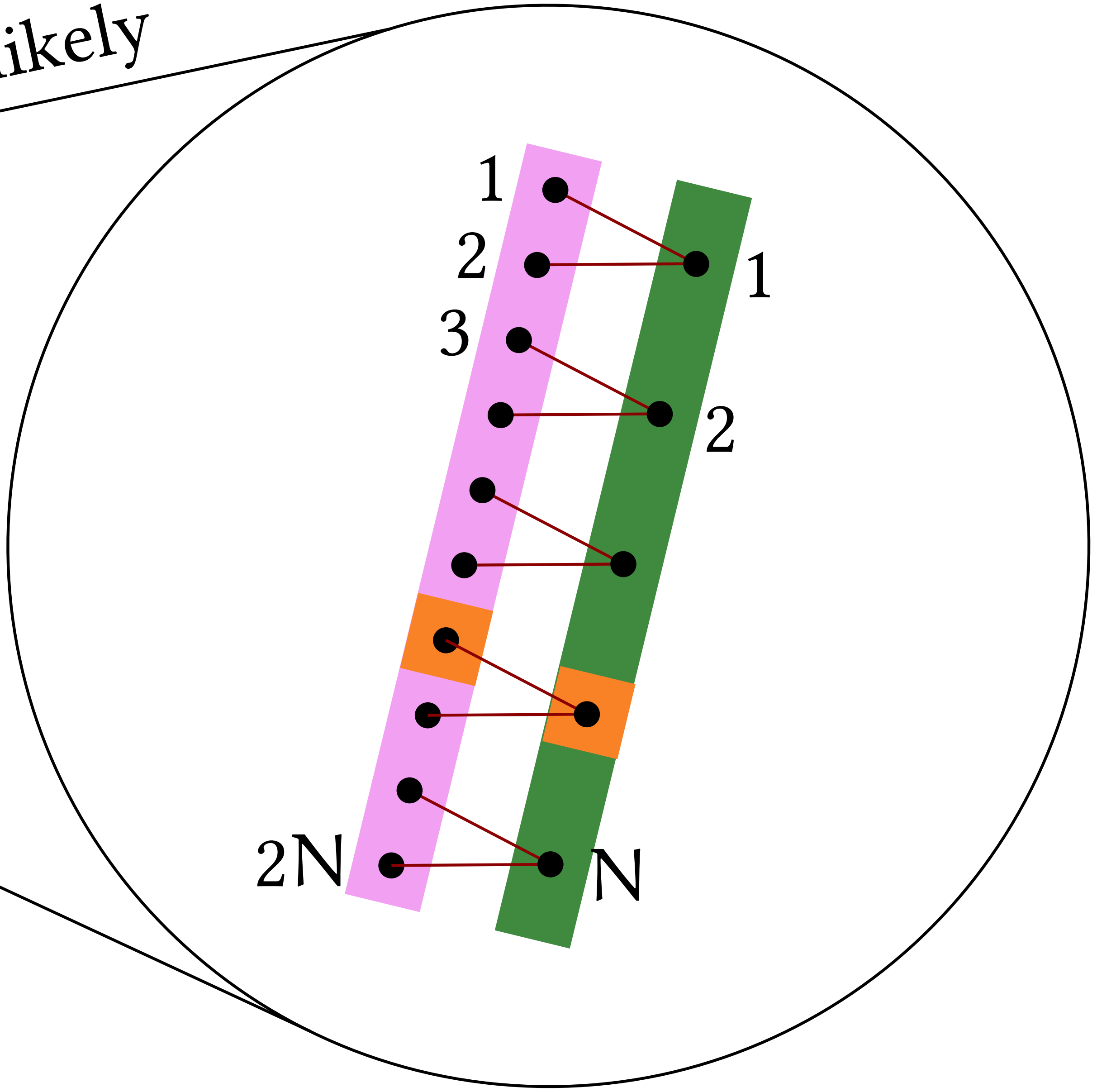
Rich 2-to-1 Games



any pairing of [2N] is equally likely



labeling $(V \to [2N]) \cup (U \to [N])$

Equivalent to UGC on unsatisfiable instances

Conjecture: [BKM'21]

NP-hard to approximately solve any instances

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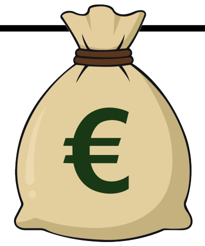
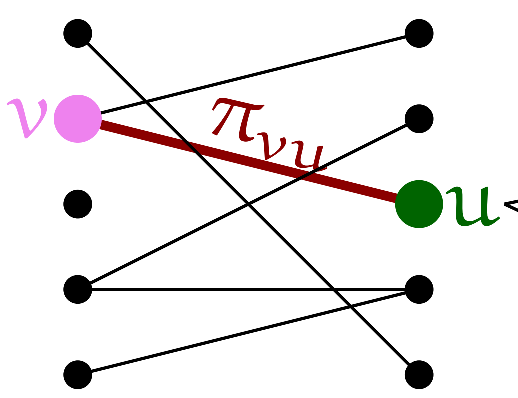


Rich 2-to-1 Games



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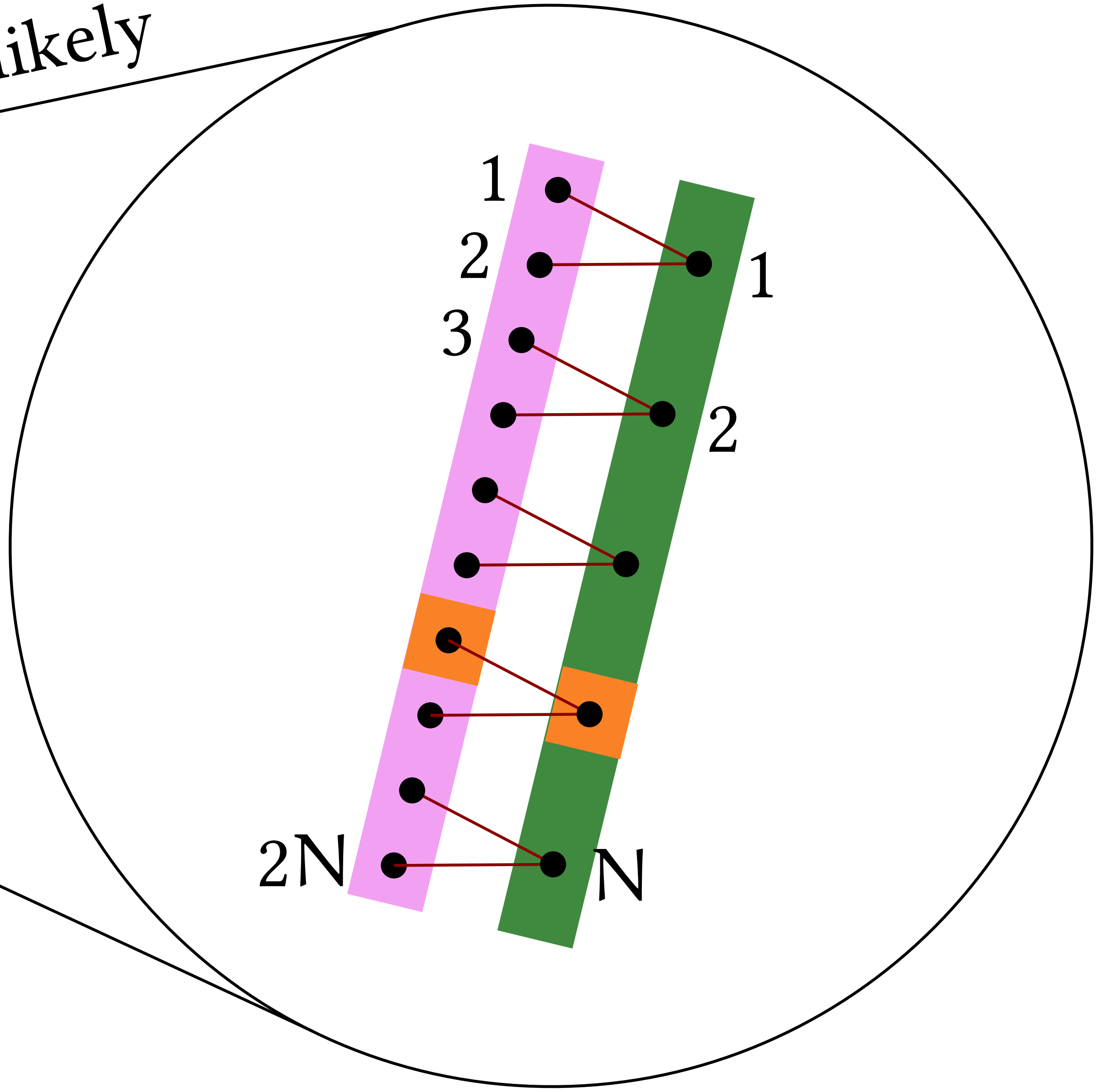

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How does $R^{(c)}_{<k:\ell}$ look like?

$\underline{R^{(2)}_{<4:5}}$

$\underline{R^{(2)}_{<4:5}}$

11122

$\underline{R^{(2)}_{<4:5}}$

11122

11222

$\underline{R^{(2)}_{<4:5}}$

11122

$\downarrow$

11222

$\downarrow$

11221

$R^{(2)}_{<4:5}$

$$\begin{array}{c} 11122 \\ \downarrow \\ 11222 \\ \downarrow \\ 11221 \\ \downarrow \\ 12221 \end{array}$$

$\underline{R^{(2)}_{<4:5}}$

$$\begin{array}{c} 11122 \\ \downarrow \\ 11222 \\ \downarrow \\ 11221 \\ \downarrow \\ 12221 \\ \downarrow \\ 12211 \end{array}$$

$R^{(2)}_{<4:5}$

11122

$\downarrow$

11222

$\downarrow$

11221

$\downarrow$

12221

$\downarrow$

12211

$\downarrow$

22211

$\underline{R^{(2)}_{<4:5}}$

$$\begin{array}{c} 11122 \\ \downarrow \\ 11222 \\ \downarrow \\ 11221 \\ \downarrow \\ 12221 \\ \downarrow \\ 12211 \\ \downarrow \\ 22211 \\ \downarrow \\ 22111 \end{array}$$

Reconfigurability

$$\underline{R^{(2)}_{<4:5}}$$

$$11{\color{orange}1}22$$
$$\downarrow$$
$$1122{\color{orange}2}$$
$$\downarrow$$
$$1{\color{orange}1}221$$
$$\downarrow$$
$$122{\color{orange}2}1$$
$$\downarrow$$
$${\color{orange}1}2211$$
$$\downarrow$$
$$22{\color{orange}2}11$$
$$\downarrow$$
$$22111$$

Reconfigurability

$$\underline{R^{(2)}_{<4:5}}$$

$$\begin{array}{c} 11122 \\ \downarrow \\ 11222 \\ \downarrow \\ 11221 \\ \downarrow \\ 12221 \\ \downarrow \\ 12211 \\ \downarrow \\ 22211 \\ \downarrow \\ 22111 \end{array}$$

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$\rightarrow$ Universal algebra
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$$LO_\ell^{(c)} = \left\{(x_1, \ldots, x_\ell) \in [c]^\ell \;\middle|\; \exists \text{unique max}\right\}$$

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☞ Understand how reconfigurability connects analytical, topological, and algebraic methods